3A Surface elements and the first fundamental form

3.1. Definition Let $U \subset \mathbb{R}^2$ be an open set. A parametrized surface element is an immersion

$$f : U \rightarrow \mathbb{R}^3.$$ 

$f$ is also called a parametrization, the elements of $U$ are called the parameters, and their images under $f$ are called points. The Cartesian coordinates in $U$ are then mapped by $f$ onto coordinate lines in the surface element; see Figure 3.1 for such a grid of coordinate lines.

A (non-parametrized) surface element is an equivalence class of parametrized surface elements, where two parametrizations $f : U \rightarrow \mathbb{R}^3$ and $\tilde{f} : \tilde{U} \rightarrow \mathbb{R}^3$ are viewed as being equivalent if there is a diffeomorphism $\varphi : \tilde{U} \rightarrow U$ such that $f = \tilde{f} \circ \varphi$.

Sometimes one also speaks of regular surface elements if the rank of the map $f$ is maximal, i.e., if $f$ is an immersion. If there turn out to be points, however, where the rank is not maximal, one speaks of singular points or singularities.

Similarly, one defines a hypersurface element in $\mathbb{R}^{n+1}$ by means of an immersion of an open subset $U$ of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$, and even more generally a $k$-dimensional surface element in $\mathbb{R}^n$.

Remarks:
1. The classical notion of a parametrization is given by a triple of functions $x, y, z$ in Cartesian coordinates

$$f(u, v) = (x(u, v), y(u, v), z(u, v)) \in \mathbb{R}^3.$$ 

The parameter $(u, v)$ is mapped here to the point $(x, y, z)$. The property of $f = f(u, v)$ of being an immersion is equivalent to the property that the vectors $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are linearly independent at every point. These span the tangent plane. The orthogonal complement to this plane is the (1-dimensional) normal space.

We introduce the following notations for a parametrized surface element $f : U \rightarrow \mathbb{R}^3, u \in U, p = f(u)$:

- $T_u U$ is the tangent space of $U$ at $u$, $T_u U = \{u\} \times \mathbb{R}^2$.
- $T_p \mathbb{R}^3$ is the tangent space of $\mathbb{R}^3$ at $p$, $T_p \mathbb{R}^3 = \{p\} \times \mathbb{R}^3$.
- $T_u f$ is the tangent plane of $f$ at $p$, $T_u f := Df(u)(T_u U) \subset T_{f(u)} \mathbb{R}^3$.
- $\perp_u f$ is the normal space of $f$ at $p$, $T_u f \oplus \perp_u f = T_{f(u)} \mathbb{R}^3$.

The elements of $T_u f$ are called tangent vectors and the elements of $\perp_u f$ are called normal vectors. Similarly, the vectors in the tangent space $T_p M$ of a submanifold $M \subset \mathbb{R}^3$ are called tangent vectors (to $M$ at $p$) and the elements of the subspace $\perp_p M$ are called normal vectors (cf. 1.7, 1.8).

2. A two-dimensional submanifold of $\mathbb{R}^3$ (cf. Def. 1.5) can be locally described as a surface element. The parametrization in this case however is far from being unique. For example, certain parts of the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ can be parametrized by

$$(u, v) \mapsto (u, v, \pm \sqrt{1 - u^2 - v^2}), \quad u^2 + v^2 < 1,$$

or by the so-called spherical coordinates (cf. Figure 3.2)

$$(\varphi, \theta) \mapsto (\cos \varphi \cos \theta, \sin \varphi \cos \theta, \sin \varphi), \quad 0 < \varphi < 2\pi, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$ 

Figure 3.2: Sphere with spherical coordinates
3 The Local Theory of Surfaces

3. The graph of an arbitrary real-valued differentiable function \( h(u,v) \) can be viewed as the image of the immersion

\[
f(u,v) := (u,v,h(u,v)).
\]

Here \( \frac{\partial}{\partial u} = (1,0,h_u), \frac{\partial}{\partial v} = (0,1,h_v) \) are always linearly independent. Conversely, by Theorem 1.4 every two-dimensional submanifold (and also every surface element) can locally be described by the graph of a function, if the coordinates are chosen appropriately.

4. As to what exactly is to be understood under a surface in the large, there are several different possibilities for how this is precisely defined. A two-dimensional submanifold certainly also can be viewed as a global surface. This excludes self-intersections. This matter can only be completely clarified upon introduction of the notion of an (abstract) two-dimensional manifold, which we postpone until Section 5.1. A surface in the large will then be defined as an immersion of a two-dimensional manifold in \( \mathbb{R}^3 \).

![Figure 3.3: Torus of revolution](image)

**Example:** A torus of revolution is defined as a surface element by

\[
f(u,v) = ((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u),
\]

\[0 < u, v < 2\pi, 0 < b < a.
\]

Because of the periodicity of sine and cosine, this parametrization closes after a period of \( 2\pi \) in every coordinate direction, if one goes beyond the interval \( u,v \in (0,2\pi) \). One then obtains the (two-dimensional) torus as a global submanifold, cf. Figure 3.3. The latter is given for example by the equation \((a^2 - b^2 + x^2 + y^2 + z^2)^2 = 4a^2(x^2 + y^2)\).

3A Surface elements and the first fundamental form

3.2. Definition (First fundamental form)

We denote by \( \langle \cdot, \cdot \rangle \) the Euclidean inner product in \( \mathbb{R}^3 \) as well as in every tangent space \( T_p \mathbb{R}^3 \). The first fundamental form \( I \) of a surface element (resp. of a two-dimensional submanifold) is just the restriction of \( \langle \cdot, \cdot \rangle \) to all tangent planes \( T_p \mathbb{R}^3 \) (resp. \( T_p M \)), i.e.,

\[
I(X,Y) := \langle X,Y \rangle
\]

for any two tangent vectors \( X, Y \in T_p U \) (resp. \( T_p M \)).

In an explicit parametrization one can view this also as a symmetric bilinear form on \( T_p U \), that is, as a mapping

\[
T_p U \times T_p U \ni (V,W) \mapsto \left( Df|_u(V), Df|_u(W) \right).
\]

This is also often referred to as the first fundamental form, and is denoted by \( I \) or \( Df \cdot Df \) or \( df \cdot df \) or \( df \otimes df \).

**Remarks:**

In coordinates \( f(u,v) = (x(u,v), y(u,v), z(u,v)) \), the first fundamental form is described by the following symmetric, positive definite matrix:

\[
(g_{ij}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} I(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial u}) & I(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}) \\ I(\frac{\partial f}{\partial v}, \frac{\partial f}{\partial u}) & I(\frac{\partial f}{\partial v}, \frac{\partial f}{\partial v}) \end{pmatrix}
\]

\[
= \begin{pmatrix} \langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u} \rangle & \langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \rangle \\ \langle \frac{\partial f}{\partial v}, \frac{\partial f}{\partial u} \rangle & \langle \frac{\partial f}{\partial v}, \frac{\partial f}{\partial v} \rangle \end{pmatrix}.
\]

In case the parametrization \( f \) is \( k \)-times continuously differentiable, the matrix \( (g_{ij}) \) of the first fundamental form is \((k-1)\)-times continuously differentiable. This matrix \( (g_{ij}) \) is also called the metric tensor or measure tensor, because it can be interpreted as a tensor (cf. Section 6A) which determines the metric properties (that is the notion of measure here). More precisely, one can also write

\[
\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix},
\]
3 The Local Theory of Surfaces

to indicate that $E, F, G$ are functions of $u$ and $v$. In terms of these parameters, one often writes the first fundamental form as a quadratic differential:

$$ds^2 = Edu^2 + 2F dudv + G dv^2;$$

d$s^2$ (or $ds$) is also called the element of arc length or the arc element or the line element. It is in fact true that for a curve $c(t) = f(u(t), v(t))$, the expression

$$\sqrt{E \left(\frac{du}{dt}\right)^2 + 2F \frac{du}{dt} \frac{dv}{dt} + G \left(\frac{dv}{dt}\right)^2}$$

is equal to the length $||\dot{c}||$ of the tangent vector $\dot{c}(t)$, which is easily seen by applying the chain rule: $\dot{c} = f_u\dot{u} + f_v\dot{v}$ implies $\langle \dot{c}, \dot{c} \rangle = (f_u, f_u)\dot{u}^2 + 2(f_u, f_v)\dot{u}\dot{v} + (f_v, f_v)\dot{v}^2 = Edu^2 + 2F dudv + G dv^2$.

Note that every regular curve $c$ whose image is contained in $f(U)$ can be written as $c(t) = f(\gamma(t))$ with a regular curve $\gamma$ whose image is contained in $U$, a fact we have used here. For this it is sufficient to set (locally at first) $\gamma(t) = f^{-1}(c(t))$.

The first fundamental form $I$ can be clearly distinguished from the Euclidean inner product on $\mathbb{T}_u U$. In symbolic notation, in which $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ denotes the standard basis of the tangent space $\mathbb{T}_u U$, the inner product is always given by the following matrix:

$$\begin{pmatrix}
\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle & \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \rangle \\
\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \rangle & \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \rangle
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.$$

Compare this with the spherical coordinates

$$f(\varphi, \theta) = (\cos \varphi \cos \theta, \sin \varphi \cos \theta, \sin \theta)$$

on the sphere and the properties of the length function there. The first fundamental form is:

$$\begin{pmatrix}
E & F \\
F & G
\end{pmatrix} = \begin{pmatrix}
\cos^2 \theta & 0 \\
0 & 1
\end{pmatrix}.$$

In $U$ the length of an interval determined by the parameter values $\theta = \varphi_0, 0 \leq \varphi \leq \pi$ is always equal to $\pi$, while the length of the image curve in $f(U)$ is equal to $\pi \cos \varphi_0$. This factor in which the lengths differ, $\cos \theta$, occurs explicitly in the matrix of the first fundamental form.

3.3. Lemma  The matrix of the first fundamental form behaves as follows under a transformation of the parameters $\tilde{f} = f \circ \varphi$ (here $D\varphi$ denotes the Jacobi matrix of $\varphi$):

$$(\tilde{g}_{ij}) = (D\varphi)^T (g_{ij}) (D\varphi).$$

Proof: The equation $(\tilde{g}_{ij}) = (D\tilde{f})^T (D\tilde{f})$ results easily from the matrix multiplication of the corresponding matrices; compare exercise 1 at the end of the chapter. With this we can calculate

$$(\tilde{g}_{ij}) = (D\tilde{f})^T (D\tilde{f}) = (Df \circ D\varphi)^T (Df \circ D\varphi) =$$

$$(D\varphi)^T (Df) (Df) (D\varphi) = (D\varphi)^T (g_{ij}) (D\varphi).$$

The determinant of the first fundamental form plays an important role in the integration of functions which are defined on surface elements (so-called surface integrals). We provide here the following definition. For more details as well as the rule for substitutions we refer the reader to [26], Chapter XX, and [27].

3.4. Definition (Surface integral)

Let $f : U \to \mathbb{R}^3$ be a surface element, and suppose that $f$ is injective, viewed as a map. Let $\alpha$ be a continuous, real-valued function which is defined on all of $f(U)$. For every compact subset $Q \subset U$, the expression

$$\iint_{f(Q)} \alpha \, dA = \iint_{Q} (\alpha \circ f)(u,v)\sqrt{\text{Det}(g_{ij})} \, du \, dv$$

is well-defined, and is called a surface integral. For $\alpha \equiv 1$, one just gets the surface area. The injectivity of $f$ can be weakened to the assumption that no open set is covered more than once. In that case one would have to count the contribution of this set to the integral with a corresponding multiplicity.

Remarks: One can similarly define an integral for integrable functions on measurable subsets of $U$, for example the Lebesgue integral.
An integral is invariant under transformations of the parameter according to Lemma 3.3. More precisely one has for \( \tilde{f} = f \circ \varphi, Q = \varphi(Q), (u, v) = \varphi(\tilde{u}, \tilde{v}) \) the substitution rule

\[
\iint_{f(Q)} \alpha \, dA = \iint_{Q} (\alpha \circ \tilde{f})(\tilde{u}, \tilde{v}) \sqrt{\text{Det}(g_{ij})} \, d\tilde{u}d\tilde{v} = \iint_{Q} (\alpha \circ \tilde{f})(\tilde{u}, \tilde{v}) \text{Det}D\varphi \sqrt{\text{Det}(g_{ij})} \, d\tilde{u}d\tilde{v} = \iint_{Q} (\alpha \circ f)(u, v) \sqrt{\text{Det}(g_{ij})} \, du dv = \iint_{f(Q)} \alpha \, dA.
\]

\( g = \text{Det}(g_{ij}) \) is also called the Gram determinant. \( \sqrt{g} \) describes the infinitesimal distortion of \( f \), which is made quite explicit by expressions like \( dA = \sqrt{g} \, du dv \). The symbol \( dA \) is meant to remind one of “area” (element of surface area). Moreover, one has

\[ g = \left\| \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \right\|^2, \]

where \( \times \) denotes the cross product or vector product in \( \mathbb{R}^3 \). (Note that in the book [1], \( \frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial v} \) is written instead of \( \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \).) The surfaces with the minimal possible surface area (with some fixed boundary) play an important role in differential geometry and analysis; see Section 3D for more details.

### 3.5. Definition (Vector fields along \( f \))

For a surface \( f: U \to \mathbb{R}^3 \) we call a map \( X: U \to \mathbb{R}^3 \) a vector field along \( f \). In this definition, we view the vector \( X(u) \) for every \( u \in U \) as a vector at the point \( p = f(u) \). In full mathematical rigor we would have to view \( X \) as a map from \( U \) to the tangent bundle \( T\mathbb{R}^3 \), where the parameter \( u \) gets mapped to \( (f(u), X(u)) \in T_f(u)\mathbb{R}^3 \). One also refers to this situation by saying that \( f(u) \) is the position vector and \( X(u) \) is the directional vector. The idea is that the directional vector \( X(u) \) is based at the point \( p = f(u) \), and then (viewed quite formally) this vector together with \( p \) defines an element \( (p, X(u)) \in T_p\mathbb{R}^3 \equiv \mathbb{R}^3 \), compare Definition 1.6.

Similarly, \( X \) is called tangential (resp. normal), if for every \( u \in U \) one has the relation \( X(u) \in T_u f \) (resp. \( X(u) \in \perp_u f \)) (note that \( T_u f \oplus \perp_u f = T_{f(u)}\mathbb{R}^3 \)).

### 3A Surface elements and the first fundamental form

A tangential vector field can always be uniquely written (with \( u = (u_1, u_2) \)) as

\[ X(u) = \alpha(u) \left. \frac{\partial f}{\partial u_1} \right|_u + \beta(u) \left. \frac{\partial f}{\partial u_2} \right|_u, \]

while a normal vector field can always be uniquely written in the form

\[ X(u) = \gamma(u) \left. \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right|_u. \]

\( X \) is said to be continuous (resp. differentiable), if \( \alpha, \beta \) and \( \gamma \) are all continuous (resp. differentiable).

**EXAMPLES:**

1. On the cylinder \( f(\varphi, x) = (\cos \varphi, \sin \varphi, x) \) the vector field

\[ X(\varphi, x) := (-\sin \varphi, \cos \varphi, x_0) \]

with constant \( x_0 \) is a tangential vector field and at the same time a tangent vector to the family of lines of inclination \( t \mapsto (\cos t, \sin t, x_0 t + c) \) with parameter \( c \) (cf. Figure 3.4).

2. Starting with some (variable) point, the unit vector

\[ \nu = \pm \left( \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right) / \left\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\| \]

is a normal vector field. The unit normal \( \nu \) can also be viewed as a map

\[ \nu: U \to S^2 \subset \mathbb{R}^3. \]

Here the vector is based at the origin. This map, the Gauss map, is of great importance in the theory of surfaces, because it determines the second fundamental form and through this also the curvature, as we will see in 3.8–3.10.

### 3.6. Definition (Orientability)

A submanifold of \( \mathbb{R}^3 \) is called orientable, if one can cover it by images of parametrized surface elements (charts in an atlas, cf. [28]) with the following property: all the Jacobi determinants of the local coordinate transformations are positive. The choice of such a cover by means of
3 The Local Theory of Surfaces

3A Surface elements and the first fundamental form

3.7. Lemma A two-dimensional submanifold $M$ of $\mathbb{R}^3$ is orientable if and only if there is a continuous unit normal vector field $\nu$ on $M$, i.e., a globally defined, continuous mapping

$$M \ni p \mapsto \nu(p) \in \mathbb{R}^3.$$

In local coordinates $f(u_1, u_2)$ the vector field $\nu$ can be expressed as follows:

$$\nu = \pm \left( \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right) / \left\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\|.$$

Figure 3.4: A tangential vector field on a cylinder

For a surface element $f: U \rightarrow \mathbb{R}^3$ the choice of an orientation can also be expressed as the choice of the order of the two tangent vectors

$$\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}.$$

A change of parameters with positive Jacobi determinant would preserve the orientation, although in general it would not preserve this particular two-frame. If the orientation of $\mathbb{R}^3$ is considered as being fixed, then the sign of the determinant

$$\text{Det} \left( \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}, \nu \right)$$

gives information about the orientation of the surface element in terms of a given unit normal $\nu$. From this we obtain the following lemma:

Example: The Möbius strip is an example of a non-orientable surface.

The image of the parametrized surface element $f: \mathbb{R} \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$ with

$$f(u, v) = \left( \sin u + v \sin \frac{u}{2} \sin u, \cos u - v \sin \frac{u}{2} \cos u, v \cos \frac{u}{2} \right)$$

is closed in the $u$-direction after one revolution $0 \leq u \leq 2\pi$, but in such a manner that a chosen unit normal vector for $u = 0$ is continuously transformed to the opposite unit normal at $u = 2\pi$. This surface is called the Möbius strip, named after the German mathematician A. Möbius. From this it follows that the image of $f$, viewed as a submanifold, is not orientable. We note also that this surface is a ruled surface in the sense of Definition 3.20 below, since the $v$-curves are segments of lines.

Figure 3.5: Möbius strip