Quadratic forms

1. Symmetric matrices

An $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ with entries on $\mathbb{R}$ is called symmetric if $A = A^T$, that is, if $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$. We denote by $S_n(\mathbb{R})$ the set of all $n \times n$ symmetric matrices with coefficients in $\mathbb{R}$.

**Remark 1.1.** Note that if $B$ is an $n \times n$ matrix with entries in $\mathbb{R}$ (not necessarily symmetric), then for any vectors $x, y \in \mathbb{R}^n$ (thought of as column-vectors), we have

$$y \cdot Ax = y^T A x = x^T A^T y.$$

Here we think of the dot-product $y \cdot Ax$ as a $1 \times 1$ matrix.

We summarize the properties of symmetric matrices in the following statement:

**Theorem 1.2.** Let $n \geq 1$ be an integer and let $A \in S_n(\mathbb{R})$ be a symmetric matrix.

Then the following hold:

1. For any vectors $x, y \in \mathbb{R}^n$ we have $Ax \cdot y = x \cdot Ay$.
2. The characteristic polynomial $p_A(t)$ of $A$ completely factors out over $\mathbb{R}$, that is, there exist (not necessarily distinct) $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $p_A(t) = (t - \lambda_1) \ldots (t - \lambda_r)$.
3. If $\lambda \neq \lambda'$ are distinct eigenvalues of $A$ and $x, x' \in \mathbb{R}^n$ are nonzero eigenvectors corresponding to $\lambda, \lambda'$ accordingly (that is, $Ax = \lambda x$, $Ax' = \lambda' x'$) then $x \cdot x' = 0$.
4. If $\lambda$ is an eigenvalue of $A$ with multiplicity $k \geq 1$ in $p_A(t)$, then there exist $k$ linearly independent eigenvectors of $A$ corresponding to $\lambda$, and the eigenspace $E_{\lambda} = \{x \in \mathbb{R}^n | Ax = \lambda x\}$ has dimension $k$.
5. There exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$.

Let $p_A(t) = (t - \lambda_1) \ldots (t - \lambda_r)$ be the factorization of $p_A(t)$ given by part (2) above, and let $v_1, \ldots, v_r \in \mathbb{R}^n$ be an orthonormal basis of $\mathbb{R}^n$ such that $Av_i = \lambda_i v_i$ (such a basis exists by part (5)). Let $C = [v_1 | \ldots | v_n]$ be the $n \times n$ matrix with columns $v_1, \ldots, v_n$.

Then $C \in O(n)$ and

$$C^{-1} A C = \text{Diag}(\lambda_1, \ldots, \lambda_n).$$

Here

$$\text{Diag}(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix}
\lambda_1 & 0 & 0 & \ldots & 0 \\
0 & \lambda_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_n
\end{bmatrix}$$

Thus symmetric matrices are diagonalizable, and the diagonalization can be performed by means of an orthogonal matrix.
2. Quadratic forms

**Definition 2.1** (Quadratic form). Let V be a vector space over \( \mathbb{R} \) of finite dimension \( n \geq 1 \).

A **quadratic form** on V is a bilinear map \( Q : V \times V \to \mathbb{R} \) such that \( Q \) is symmetric, that is, \( Q(v, w) = Q(w, v) \) for all \( v, w \in V \).

Saying that \( Q \) is bilinear means that

\[
Q(c_1 v_1 + c_2 v_2, w) = c_1 Q(v_1, w) + c_2 Q(v_2, w)
\]

and

\[
Q(v, c_1 w_1 + c_2 w_2) = c_1 Q(v, w_1) + c_2 Q(v, w_2)
\]

for all \( c_1, c_2 \in \mathbb{R} \) and \( v, v_1, v_2, w, w_1, w_2 \in V \).

To every quadratic form \( Q : V \times V \to \mathbb{R} \) we also associate the “quadratic” function \( q : V \to \mathbb{R} \) defined as \( q(v) := Q(v, v) \) for \( v \in V \). We will see in Proposition 2.3 below that the quadratic form \( Q \) can be uniquely recovered from \( q \).

**Example 2.2.** We have:

1. The standard dot-product \( : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, (x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \), is a quadratic form on \( \mathbb{R}^n \).
2. Let \( A \) be an \( n \times n \) matrix with entries in \( \mathbb{R} \). Define

\[
Q_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}
\]

as \( Q_A(x, y) := y^T A x = y \cdot A x = A y \cdot x = \sum_{i,j=1}^n a_{ij} y_i x_j \).

Then \( Q_A \) is a quadratic form on \( \mathbb{R}^n \).
3. Let \( A \) be an \( n \times n \) matrix with entries in \( \mathbb{R} \) and let \( B = \frac{1}{2}(A + A^T) \) (so that \( B = B^T \) and \( B \) is symmetric). Then \( Q_A = Q_B \), that is, for all \( x, y \in \mathbb{R}^n \) we have \( Q_A(x, y) = Q_B(x, y) \).
4. Let \( V \) be a vector space with a basis \( \mathcal{E} = e_1, \ldots, e_n \) and let \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) be real numbers.

Define \( Q : V \times V \to \mathbb{R} \) as \( Q(x, y) = \sum_{i=1}^n \lambda_i y_i x_i \), where \( x = \sum_{i=1}^n x_i e_i \) and \( y = \sum_{i=1}^n y_i e_i \) are arbitrary vectors in \( V \).

Then \( Q \) is a quadratic form on \( V \).
5. Let \( V \) be a vector space with a basis \( \mathcal{E} = e_1, \ldots, e_n \) and let \( A \in S_n(\mathbb{R}) \) be a symmetric matrix.

Define \( Q_{A, \mathcal{E}} : V \times V \to \mathbb{R} \) as

\[
Q_{A, \mathcal{E}}(x, y) := \sum_{i,j=1}^n a_{ij} y_i x_j
\]

where \( x = \sum_{i=1}^n x_i e_i \) and \( y = \sum_{i=1}^n y_i e_i \) are arbitrary vectors in \( V \).

Then \( Q_{A, \mathcal{E}} \) is a quadratic form on \( V \).

**Proposition 2.3.** The following hold:

1. For any quadratic form \( Q \) on \( \mathbb{R}^n \) there exists a unique symmetric matrix \( A \in S_n(\mathbb{R}) \) such that \( Q = Q_A \).
(2) Let $V$ be a vector space of dimension $n$ with a basis $\mathcal{E} = e_1, \ldots, e_n$.

Let $Q : V \times V \to \mathbb{R}$ be a quadratic form on $V$. Then there exists a unique symmetric matrix $A = (a_{ij})_{ij} \in S_n(\mathbb{R})$ such that $Q = Q_{A,\mathcal{E}}$.

Moreover $a_{ij} = Q(e_i, e_j)$ for all $1 \leq i, j \leq n$.

(3) Let $V, Q, \mathcal{E}$ and $A$ be as in (2) above. Then for $i = 1, \ldots, n$ we have $a_{ii} = Q(e_i, e_i)$ and for any $1 \leq ij \leq n$, $i \neq j$ we have

$$a_{ij} = \frac{1}{2}(Q(e_i + e_j, e_i + e_j) - Q(e_i, e_i) - Q(e_j, e_j)).$$

Thus part (3) of Proposition 2.3 shows that if $Q : V \times V$ is a quadratic form, then we can recover $Q$ from knowing the values $Q(x, x)$ for all $x \in V$.

Proposition 2.3 describes all quadratic forms on $V$ in terms of their associated symmetric matrices, once we choose a basis $\mathcal{E}$ of $V$. It is useful to know how these matrices change when we change the basis of $V$.

**Proposition 2.4.** Let $V$ be a vector space of dimension $n$ and let $Q : V \times V \to \mathbb{R}$ be a quadratic form on $V$.

Let $\mathcal{E} = e_1, \ldots, e_n$ and $\mathcal{E}' = e'_1, \ldots, e'_n$ be two bases of $V$. Let $A, A'$ be symmetric matrices such that $Q = Q_{A,\mathcal{E}} = Q_{A',\mathcal{E}'}$.

Then

$$A' = C^T AC,$$

where $C$ is the transition matrix from $\mathcal{E}$ to $\mathcal{E}'$, that is $e'_j = \sum_{i=1}^n c_{ij} e_j$.

Together with Theorem 1.2, Proposition 2.4 implies:

**Corollary 2.5.** Let $V$ be a vector space of dimension $n$ and let $Q : V \times V \to \mathbb{R}$ be a quadratic form on $V$. Then there exist a basis $\mathcal{E} = e_1, \ldots, e_n$ and real numbers $\lambda_1 \leq \cdots \leq \lambda_n$ such that $Q = Q_{A,\mathcal{E}}$ with $A = \text{Diag}(\lambda_1, \ldots, \lambda_n)$. This means that $Q(x, y) = \sum_{i=1}^n \lambda_i x_i y_i$, where $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{i=1}^n y_i e_i$ are arbitrary vectors in $V$.

The values $\lambda_1, \ldots, \lambda_n$ in Corollary 2.5 are not uniquely determined by $Q$. However, it turns out that the number $n_+$ of $i \in \{1, \ldots, n\}$ such that $\lambda_i > 0$, the number $n_-$ of $i \in \{1, \ldots, n\}$ such that $\lambda_i < 0$ and the number $n_0$ of $i \in \{1, \ldots, n\}$ such that $\lambda_i = 0$ are uniquely determined by $Q$. The triple $(n_0, n_+, n_-)$ is called the signature of $Q$. Note that by definition, $n_0 + n_+ + n_- = n$.

**Proposition 2.6.** Let $V$ be a vector space over $\mathbb{R}$ of finite dimension $n \geq 1$.

Then the following are equivalent:

1. The form $Q$ is positive-definite, that is, $Q(x, x) > 0$ for every $x \in V, x \neq 0$.
2. We have $n_+ = n$ and $n_0 = n_- = 0$.

**Remark 2.7.** Thus we see that if $Q$ is a quadratic form on $V$ with $Q = Q_{A,\mathcal{E}}$ for some basis $\mathcal{E}$ of $V$ and a symmetric matrix $A \in S_n(\mathbb{R})$ with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then $Q$ is positive-definite if and only if $\lambda_i > 0$ for $i = 1, \ldots, n$ (in which case we also have $\det(A) = \lambda_1 \cdots \lambda_n > 0$).
Proposition 2.6 implies that if \( Q \) is a positive-quadratic form on a vector-space \( V \) of finite dimension \( n \geq 1 \), then for every nonzero linear subspace \( W \) of \( V \) the restriction of \( Q \) to \( W, Q : W \times W \to \mathbb{R} \), is again a positive-definite quadratic form.

3. The Case of Dimension 2

Let \( V \) be a vector space over \( \mathbb{R} \) of dimension 2. Let \( \mathcal{E} = e_1, e_2 \) be a basis of \( V \) and let \( Q : V \times V \to \mathbb{R} \) be a quadratic form on \( V \).

Then \( Q = Q_{A,\mathcal{E}} \) where

\[
A = \begin{bmatrix}
E & F \\
F & G
\end{bmatrix}
\]

where \( E = Q(e_1, e_1), F = Q(e_1, e_2) = Q(e_2, e_1) \) and \( G = Q(e_2, e_2) \). For \( x = x_1e_1 + x_2e_2 \in V \) and \( y = y_1e_1 + y_2e_2 \in V \) we have

\[
Q(x, y) = \sum_{i,j} = 1^n a_{ij} y_i x_j = Ex_1y_1 + Fx_1y_2 + Fx_2y_1 + Gx_2y_2
\]

and \( Q(x, x) = Ex_1^2 + 2Fx_1x_2 + Gx_2^2 \).

**Example 3.1.** (a) Let \( Q : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( Q((x_1, x_2), (y_1, y_2)) = x_1y_1 + 3x_1y_2 + 3x_2y_1 + x_2y_2 \). Thus for the standard unit basis \( \mathcal{E} = e_1, e_2 \) of \( \mathbb{R}^2 \) we have \( Q = Q_{A,\mathcal{E}} \) where

\[
A = \begin{bmatrix}
1 & 3 \\
3 & 1
\end{bmatrix}
\]

so that \( E = G = 1, F = 3 \). The characteristic polynomial of \( A \) is \( \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) \), with roots \( \lambda_1 = 4, \lambda_2 = -2 \). Thus the signature of \( Q \) is \( n_+ = n_- = 1, n_0 = 0 \), and the quadratic form \( Q \) is not positive-definite.

(b) Let \( Q : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( Q((x_1, x_2), (y_1, y_2)) = x_1y_1 + 2x_1y_1 + 2x_2y_1 + 7x_2y_2 \). Thus for the standard unit basis \( \mathcal{E} = e_1, e_2 \) of \( \mathbb{R}^2 \) we have \( Q = Q_{A,\mathcal{E}} \) where

\[
A = \begin{bmatrix}
1 & 2 \\
2 & 7
\end{bmatrix}
\]

so that \( E = G = 1, F = 3 \). The characteristic polynomial of \( A \) is \( \lambda^2 - 8\lambda + 3 \), with roots \( \lambda_1 = 4 - \sqrt{13}, \lambda_2 = 4 + \sqrt{13} \). Since \( \lambda_1 > 0, \lambda_2 > 0 \), the signature of \( Q \) is \( n_+ = 2, n_0 = n_- = 0 \), and the quadratic form \( Q \) is positive-definite.

3.1. Riemannian metrics. A Riemannian metric \( g \) on \( \mathbb{R}^n \) is a function that, to every \( p \in \mathbb{R}^n \), associates a positive-definite quadratic form \( g(p) \) on \( T_p\mathbb{R}^n \).

Thus for every \( p \in \mathbb{R}^n \) \( g|_p \) has the form \( g|_p = Q_{A(p)} \) for some symmetric \( n \times n \) matrix \( A(p) = (g_{ij}(p))_{ij} \), so that

\[
g_p((x_1, \ldots, x_n)_p, (y_1, \ldots, y_n)_p) = \sum_{i,j=1}^n g_{ij}(p)y_i x_j
\]
and
\[ g_p((x_1, \ldots, x_n)_p, (x_1, \ldots, x_n)_p) = \sum_{i,j=1}^{n} g_{ij}(p)x_i x_j. \]

Note that \( g_{ij}(p) = g|_p((e_i)_p, (e_j)_p) \). We think of \( g|_p \) as an “inner product” on \( T_p \mathbb{R}^n \).

In the case \( n = 2 \) the matrix \( A(p) \) is
\[ A = \begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix} \]
so that
\[ g|_p((x_1, x_2)_p, (y_1, y_2)_p) = E(p)x_1 y_1 + F(p)x_1 y_2 + F(p)x_2 y_1 + G(p)x_2 y_2 \]
and
\[ g|_p((x_1, x_2)_p, (x_1, x_2)_p) = E(p)x_1^2 + 2F(p)x_1 x_2 + G(p)x_2^2. \]

A Riemannian metric \( g \) on \( \mathbb{R}^n \) is smooth if the functions \( g_{ij} : \mathbb{R}^n \to \mathbb{R} \) are smooth for all \( 1 \leq i, j \leq n \).

Let \( g \) be a smooth Riemannian metric on \( \mathbb{R}^n \). If \( v_p \in T_p \mathbb{R}^n \), we put
\[ ||v_p||_g := \sqrt{g|_p(v_p, v_p)} \]
and call this number the \( g \)-norm of \( v_p \).

For a smooth curve \( \gamma : [a, b] \to \mathbb{R}^n \), \( \gamma(t) = (x_1(t), \ldots, x_n(t)) \), we define the length of \( \gamma \) with respect to \( g \) as
\[ s := \int_a^b ||\gamma'(t)||_g \, dt = \int_a^b \sqrt{\sum_{i,j=1}^{n} g_{ij}(\gamma(t)) \frac{dx_i}{dt} \frac{dx_j}{dt}} \, dt \]

For this reason, the Riemannian metric \( g \) is often symbolically denoted as
\[ ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j. \]