Orientation and orientability

1. Orientation on a vector space

Throughout this section let $V$ be a vector space over $\mathbb{R}$ of finite dimension $n \geq 1$. For two bases $B = b_1, \ldots, b_n$ and $B' = b'_1, \ldots, b'_n$ of $V$, the transition matrix

$$T_{B',B} = (a_{ij})_{i,j=1}^n$$

from $B$ to $B'$ is an $n \times n$ matrix where

$$b_i = \sum_{j=1}^n a_{ji} b'_j.$$

The following proposition records some basic facts about transition matrices:

**Proposition 1.1.** The following hold:

1. For any basis $B$ of $V$ we have $T_{B,B} = I_n$, the $n \times n$ identity matrix.
2. For any bases $B, B', B''$ of $V$ we have $T_{B''B'} T_{B'B} = T_{B''B}$.  
3. For any bases $B, B'$ of $V$ we have $T_{B,B'} = T_{B,B'}^{-1}$; in particular, $\det T_{B',B} \neq 0$.

**Proof.** We will give a proof of part (2) which is the central point of this proposition.

Denote $(a_{ij})_{ij} = T_{B,B'}$ and $(d_{kl})_{kl} = T_{B'',B'}$.

Thus for every $i = 1, \ldots, n$ we have

$$b_i = \sum_{j=1}^n a_{ji} b'_j$$

and for every $j = 1, \ldots, n$ we have

$$b'_j = \sum_{k=1}^n d_{kj} b''_k.$$

Therefore

$$b_i = \sum_{j=1}^n a_{ji} b'_j = \sum_{j=1}^n \sum_{k=1}^n a_{ji} d_{kj} b''_k =$$

$$\sum_{k=1}^n \left( \sum_{j=1}^n d_{kj} a_{ji} \right) b''_k = (T_{B'',B'} T_{B',B})_{ki} b''_k$$

and therefore $T_{B'',B'} T_{B',B} = T_{B'',B}$.

In view of the above proposition, the following definition is natural:

**Definition 1.2** (Bases with the same orientation). Let $B$ and $B'$ be bases of $V$. We say that $B'$ has the same orientation as $B$, and write $B' \sim B$, if $\det T_{B',B} > 0$. We $B'$ has the opposite orientation from $B$, and write $B' \not\sim B$, if $\det T_{B',B} < 0$. 


We have:

**Proposition 1.3.** The following hold:

1. The relation $\sim$ is an equivalence relation on the set $\mathcal{B}$ of all bases of $V$, that is:
   (a) For every basis $\mathcal{B}$ of $V$, $\mathcal{B} \sim \mathcal{B}$.
   (b) If $\mathcal{B}' \sim \mathcal{B}$ then $\mathcal{B} \sim \mathcal{B}'$.
   (c) If $\mathcal{B}' \sim \mathcal{B}$ and $\mathcal{B}'' \sim \mathcal{B}'$ then $\mathcal{B}'' \sim \mathcal{B}$.

2. For any basis $\mathcal{B} = b_1, \ldots, b_n$ of $V$ we have $\mathcal{B} \sim \mathcal{B}'$, where $\mathcal{B}' = -b_1, b_2, \ldots, b_n$.

3. There are exactly two distinct equivalence classes for the equivalence relation $\sim$ on $\mathcal{B}$.

**Definition 1.4** (Orientation). An orientation on $V$ is a choice of an equivalence class for an equivalence relation $\sim$ on the set $\mathcal{B}$ of all bases of $V$.

In practice, an orientation on $V$ is specified by choosing a specific basis $B = b_1, \ldots, b_n$ and taking the orientation to be the $\sim$-equivalence class of $B$. Then for any other basis $B'$ of $V$, the basis $B'$ is positively oriented with respect to this orientation if $\det T_{B',B} > 0$, and $B'$ is negatively oriented with respect to this orientation if $\det T_{B',B} < 0$.

**Definition 1.5** (Standard orientation on $\mathbb{R}^n$). For $n \geq 1$ the standard orientation on $\mathbb{R}^n$ is determined by the standard unit basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$.

We record the following basic facts about the standard orientation in dimensions 2 and 3:

**Proposition 1.6.** The following hold:

1. A basis $\mathcal{B} = b_1, b_2$ of $\mathbb{R}^2$ is positively oriented with respect to the standard orientation on $\mathbb{R}^2$ if and only if moving from $b_1$ to $b_2$ is along the angle $< \pi$ between these two vectors gives the movement in the counter-clockwise direction.

2. An orthonormal basis $\mathcal{B} = b_1, b_2, b_3$ of $\mathbb{R}^3$ is positively oriented with respect to the standard orientation on $\mathbb{R}^3$ if and only if $b_3 = b_1 \times b_2$.

3. A basis $\mathcal{B} = b_1, b_2, b_3$ of $\mathbb{R}^3$ is positively oriented with respect to the standard orientation on $\mathbb{R}^3$ if and only if $(b_1 \times b_2) \cdot b_3 > 0$.

2. Orientable surfaces

Recall that we denote the coordinates on $\mathbb{R}^2$ by $u, v$ and we denote the coordinates on $\mathbb{R}^3$ by $(x, y, z)$.

**Definition 2.1.** A surface $S \subseteq \mathbb{R}^3$ is said to be orientable if there exists a collection of coordinate patches $\{\psi_\alpha : U_\alpha \to S\}_{\alpha \in A}$ (where each $U_\alpha \subseteq \mathbb{R}^2$ is an open subset and $\psi_\alpha$ is an injective regular map) such that:

1. We have $\cup_{\alpha \in A} \psi_\alpha(U_\alpha) = S$. 

(2) For every \( \alpha, \beta \in A \) such that \( \psi_\alpha(U_\alpha) \cap \psi_\beta(U_\beta) \neq \emptyset \), we have
\[
\det D(\psi_\beta^{-1} \circ \psi_\alpha) > 0
\]
at every point where \( \psi_\beta^{-1} \circ \psi_\alpha \) is defined (here \( D(\psi_\beta^{-1} \circ \psi_\alpha) \) is the
\( 2 \times 2 \) Jacobi matrix of the map \( \psi_\beta \circ \psi_\alpha^{-1} \)).

A collection \( \{ \psi_\alpha : U_\alpha \to S \}_{\alpha \in A} \) as above is called an **orienting atlas** for \( S \).

An orienting atlas \( \{ \psi_\alpha : U_\alpha \to S \}_{\alpha \in A} \) defines an orientation on \( T_qS \) for every \( q \in S \) as follows: If \( q = \psi_\alpha(p) \) for some \( \alpha \in A \) and \( p \in U_\alpha \), we declare the basis \( (\psi_\alpha)_*(e_1)_p, (\psi_\alpha)_*(e_2)_p \) of \( T_qS \) (that is, the basis \( \frac{\partial \psi_\alpha}{\partial u}|_p, \frac{\partial \psi_\alpha}{\partial v}|_p \) of \( T_qS \)) to be positively oriented. Then a basis \( B \) of \( T_qS \) is **positively oriented** for the orientation determined by the orienting atlas \( \{ \psi_\alpha : U_\alpha \to S \}_{\alpha \in A} \) if and only if \( \det B > 0 \), where \( B = \frac{\partial \psi_\alpha}{\partial u}|_p \times \frac{\partial \psi_\alpha}{\partial v}|_p \).

Thus we have a choice of a positively oriented basis \( \frac{\partial \psi_\alpha}{\partial u}|_p, \frac{\partial \psi_\alpha}{\partial v}|_p \) of \( T_qS \) which varies continuously with the point \( q \in S \).

In practice one rarely uses the above definition of an orientation but rather works with computationally easier to handle but equivalent descriptions or orientability and orientation.

**Definition 2.2** (Outward unit normal). Let \( \{ \psi_\alpha : U_\alpha \to S \}_{\alpha \in A} \) be an orienting atlas on a surface \( S \subseteq \mathbb{R}^3 \).

For \( q = \psi_\alpha(p) \) (where \( \alpha \in A \) and \( p \in U_\alpha \)) put
\[
N_q := \frac{\frac{\partial \psi_\alpha}{\partial u}|_p \times \frac{\partial \psi_\alpha}{\partial v}|_p}{||\frac{\partial \psi_\alpha}{\partial u}|_p \times \frac{\partial \psi_\alpha}{\partial v}|_p||}
\]
Then \( ||N_q|| = 1 \), \( N_q \perp T_qS \) and for a basis \( B = b_1, b_2 \) of \( T_qS \) the basis \( B \) is positively oriented for the orientation determined by the orienting atlas \( \{ \psi_\alpha : U_\alpha \to S \}_{\alpha \in A} \) if and only if the basis \( b_1, b_2, N_q \) of \( \mathbb{R}^3 \) is positively oriented with respect to the standard orientation on \( \mathbb{R}^3 \).

The vector \( N_q \) is said to be the **outward unit normal** of \( S \) at \( q \) with respect to the orientation determined by the orienting atlas \( \{ \psi_\alpha : U_\alpha \to S \}_{\alpha \in A} \).

**Theorem 2.3.** Let \( S \subseteq \mathbb{R}^3 \) be a surface. Then the following conditions are equivalent:

1. \( S \) is orientable.
2. There exists a continuous vector field \( W : S \to \mathbb{R}^3 \) such that for every \( q \in S \) we have \( ||W(q)|| = 1 \) and \( W(q) \perp T_qS \).
3. There exists a continuous vector field \( W : S \to \mathbb{R}^3 \) such that for every \( q \in S \) we have \( ||W(q)|| \neq 0 \) and \( W(q) \perp T_qS \).
4. There exists a continuous (with respect to \( q \in S \)) choice of an orientation on \( T_qS \), that is, there exist continuous vector fields \( V_1, V_2 : S \to \mathbb{R}^3 \) such that for every \( q \in S \) the pair \( V_1(q), V_2(q) \) is a basis of \( T_qS \).
5. The surface \( S \) does not contain a copy of the Möbius band.
As a practical matter, we usually specify the orientation on $S$ by providing a continuous unit normal vector field $W$ on $S$ as in part (2) (or a non-vanishing continuous normal vector field as in part (3)) of the above theorem. Then a basis $b_1, b_2$ of $T_qS$ is positively oriented for the orientation on $S$ determined by $W$ if and only if $b_1, b_2, W(q)$ is a positively oriented basis of $\mathbb{R}^3$ with respect to the standard orientation on $\mathbb{R}^3$.

The following key fact provides a rich source of examples of orientable surfaces arising from the Implicit Function Theorem:

**Theorem 2.4.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, let $c \in \mathbb{R}$ and let $S = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = c\}$. Suppose that $S \neq \emptyset$ and that for every $q \in S$ we have $\text{grad}(f)|_q = (\frac{\partial f}{\partial x}|_q, \frac{\partial f}{\partial y}|_q, \frac{\partial f}{\partial z}|_q) \neq (0, 0, 0)$. Then:

1. $S$ is an orientable surface in $\mathbb{R}^3$.
2. For every $q \in S$ we have $\text{grad}(f)|_q \perp T_qS$.

Thus in the above situation we can use $\text{grad}(f)$ as a non-vanishing normal vector field to define an orientation on $S$, for which $N_q = \frac{\text{grad}(f)|_q}{||\text{grad}(f)|_q||}$ (where $q \in S$) is the outward unit normal. For this orientation a basis $B = b_1, b_2$ of $T_qS$ is positively oriented if and only if $b_1, b_2, \text{grad}(f)|_q$ is positively oriented with respect to the standard orientation on $\mathbb{R}^3$, that is, if and only if $(b_1 \times b_2) \cdot \text{grad}(f)|_q > 0$. 
