Chapter 4

The Intrinsic Geometry of Surfaces

The "intrinsic geometry" of surfaces refers to all those properties of a surface which only depend on the first fundamental form. Expressed more figuratively, the intrinsic geometry is the geometry which pure two-dimensional beings (the inhabitants of "flatland"\textsuperscript{1}) can recognize, without any knowledge of the third dimension. Surely angles and lengths are among these properties. The question naturally arises as to what geometric quantities are intrinsic, in particular, which of the curvature quantities are of this kind. On the one hand it is intuitively clear that a change in lengths and angles can lead to a change in the curvature. On the other hand it is not at all clear whether the first fundamental form alone is sufficient to determine the curvature.

A further problem in this connection is as follows. How can one form derivatives using only the properties of the surface itself, without reference to the ambient space? The directional derivative of scalar functions is defined in terms of difference quotients. This is no longer so for the directional derivative of vector fields. For vector fields living in Euclidean space it is sufficient to take the derivatives of the coordinate functions, since one has a constant basis. This is no longer true on an arbitrary surface. Instead, one has to form the

derivative of the basis itself, a process which is not \textit{a priori} well-defined. To alleviate these problems, one first reduces the process of taking derivatives to differentiation in the ambient space, and then studies whether or not the notion thus defined only depends on the first fundamental form.

In what follows, $U$ will denote an open set in $\mathbb{R}^n$ and $f: U \to \mathbb{R}^{n+1}$ will denote a hypersurface element. We will often just speak of a "surface element" in this context. The notions of "tangent" and "normal" will always be taken with respect to this $f$ if nothing to the contrary is stated. If the general dimension $n$ is too abstract to the reader, he or she may safely just think of $n$ as being 2, and the results are then the classical theory of the "intrinsic geometry of surfaces". However, since most of the formulas which we will give are just the same in dimension $n = 2$ as in general dimensions (in particular, the indices on the objects like $g_{ij}$, $h_{ij}$, etc.), we formulate everything in Chapter 4 for hypersurfaces in higher dimensions whenever this makes sense. This is also preparation for Chapters 5 to 8 which follow, in which higher dimensions occur out of necessity. In the discussion of special parameters in Section 4E, in the Gauss-Bonnet theorem in Section 4F, as well as in the global surface theory in Section 4G, we will return to consider the special case of dimension two in more detail.

There are good reasons to write the indices on coordinate functions as superscripts, and we shall do this throughout, writing for example $u = (u^1, \ldots, u^n)$ and $x = (x^1, \ldots, x^{n+1})$. The reason for this is the so-called Ricci calculus, as well as the Einstein summation convention. In the latter convention, summation symbols are omitted for sums over indices which are superscripts in one place and subscripts in another. This will be explained in more detail in Chapters 5 and 6, while we will explicitly write all summation signs in this chapter.

4A The covariant derivative

The analysis of the ambient space $\mathbb{R}^{n+1}$ leads to the notion of directional derivatives of functions and vector fields, as is well-known (cf. 4.1). For the theory of surfaces this has the disadvantage that even the derivative of tangential vector fields in the tangent direction
may very well have a normal component (this is just a directional derivative in space). This would leave the realm of ‘‘intrinsic geometry of the surface’’. There is a way out of this, by considering only the component of that directional derivative which is tangent to the surface (cf. 4.3). The so-called covariant derivative obtained in this manner has in addition a series of very pleasant properties. It is, for example, a property of the intrinsic geometry, cf. 4.6.

4.1. Definition and Lemma. (Directional derivative)

Let $Y$ be a differentiable vector field, defined on an open set of $\mathbb{R}^{n+1}$, and let $X$ be a fixed directional vector at some fixed point $p$ of this open set. (In other words, assume $(p, X) \in T_p \mathbb{R}^{n+1}$.) Then the expression

$$D_X Y \big|_p := D_Y \big|_p (X) = \lim_{t \to 0} \frac{1}{t} (Y(p + tX) - Y(p))$$

is called the directional derivative of $Y$ in the direction of $X$ (cf. [27], Chapter XVII, §2). Here $D_Y$ denotes the Jacobi matrix.

Furthermore, $D_X Y \big|_p$ is already uniquely defined by the value of $Y$ along an arbitrary differentiable curve $c: (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ with $c(0) = p$ and $c(0) = X$. More precisely, one has

$$D_X Y \big|_p = \lim_{t \to 0} \frac{1}{t} (Y(c(t)) - Y(p)).$$

The (vector-valued) partial derivatives of $Y$ correspond to the case $X = e_i$ with the standard basis $e_1, \ldots, e_n$, meaning that we have the equation $D_{e_i} Y = \frac{\partial Y}{\partial x_i}$. Consequently we have with $X = \sum_i X^i e_i$

$$D_X Y \big|_p = \sum_i X^i D_{e_i} Y \big|_p = \sum_i X^i \lim_{t \to 0} \frac{1}{t} (Y(p + te_i) - Y(p)).$$

Proof of this claim: By the chain rule we have

$$\lim_{t \to 0} \frac{1}{t} (Y(c(t)) - Y(c(0))) = \frac{d}{dt} \big|_{t=0} Y(c(t)) = D_Y \big|_p (X) = D_X Y \big|_p.$$
4.2. Consequence. For a (hyper-)surface element \( f : U \to \mathbb{R}^{n+1} \) let \( Y \) denote a differentiable vector field along \( f \), and let \( X \) be some fixed tangent vector to \( f \) at the point \( p = f(u) \) (see Definition 3.5).

Then, according to 4.1, the directional derivative \( D_X Y |_p \) is well-defined as a vector field along \( f \). More precisely, the following relation always holds at the point \( p = f(u) \):

\[
D_X Y |_p = D_Y |_u ((Df)^{-1}(X)) = \lim_{t \to 0} \frac{1}{t} \left( Y(u + t(Df)^{-1}(X)) - Y(u) \right).
\]

Here, \( c(t) = f(u + t(Df)^{-1}(X)) \) is a particular curve for which \( c(0) = X \). Hence we apply 4.1 to this. Note that \( Y \) is not defined at points of the surface element, but rather on the set of parameters. Thus \( Y\left(u + t(Df)^{-1}(X)\right) \) is a well-defined vector field along this curve.

The derivative in the direction of the \( i \)th coordinate \( u^i \) is nothing but the case \( X = \frac{\partial f}{\partial u^i} \). It follows that

\[
D_{\frac{\partial f}{\partial u^i}} Y |_p = \lim_{t \to 0} \frac{1}{t} \left( Y(u^1, \ldots, u^i + t, \ldots, u^n) - Y(u^1, \ldots, u^i, \ldots, u^n) \right)
\]

and in particular

\[
D_{\frac{\partial f}{\partial u^i}} \frac{\partial f}{\partial u^i} = \frac{\partial^2 f}{\partial u^i \partial u^i}.
\]

4.3. Definition. (Covariant derivative)

If \( X, Y \) are tangent to a hypersurface element \( f \), then the expression

\[
\nabla_X Y := (D_X Y)^{\text{Tang.}} = D_X Y - \langle D_X Y, \nu \rangle \nu
\]

is called the covariant derivative of \( Y \) in the direction of \( X \). If \( X, Y \) are tangent vector fields, then the covariant derivative \( \nabla_X Y \) is again a tangent vector field. The normal component of \( D_X Y \) is nothing but the second fundamental form of \( f \), since the equality

\[
\langle D_X Y, \nu \rangle = II(X, Y)
\]

holds because of the relation \( \langle Y, \nu \rangle = 0 \), and consequently \( \langle D_X Y, \nu \rangle = -\langle Y, D_X \nu \rangle \). Hence we can also write

\[
D_X Y = \nabla_X Y + II(X, Y) \nu.
\]
4A The covariant derivative

REMARK: In [1], $D$ is written instead of $\nabla$ for the covariant derivative. It is at any rate important to differentiate between the two differential operators (directional derivative and covariant derivative):

The directional derivative $D$ is defined for vector fields on the ambient Euclidean space.
The covariant derivative $\nabla$ is defined only for tangent vector fields on the hypersurface element.

For a scalar function $\varphi$ along $f$ there is only one kind of directional derivative in the direction of $X$, defined as the limit of a difference quotient and written $D_X \varphi = \nabla_X \varphi$. In addition, we can multiply such scalar functions pointwise with vector fields, with the notation $\varphi X$ for the vector field $p \mapsto (\varphi X)(p) = \varphi(p) \cdot X(p)$.

4.4. Lemma. (Properties of $D$ and $\nabla$)

(i) $D_{\varphi_1 X_1 + \varphi_2 X_2} Y = \varphi_1 D_{X_1} Y + \varphi_2 D_{X_2} Y$,
$\nabla_{\varphi_1 X_1 + \varphi_2 X_2} Y = \varphi_1 \nabla_{X_1} Y + \varphi_2 \nabla_{X_2} Y$. (linearity)

(ii) $D_X (Y_1 + Y_2) = D_X Y_1 + D_X Y_2$,
$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$. (additivity)

(iii) $D_X (\varphi Y) = \varphi D_X Y + D_X \varphi \cdot Y$,
$\nabla_X (\varphi Y) = \varphi \nabla_X Y + \nabla_X \varphi \cdot Y$. (product rule)

(iv) $D_X (Y_1, Y_2) = (D_X Y_1, Y_2) + (Y_1, D_X Y_2)$,
$\nabla_X (Y_1, Y_2) = (\nabla_X Y_1, Y_2) + (Y_1, \nabla_X Y_2)$. (compatibility with the inner product)

WARNING: For the directional derivative and the covariant derivative, commutativity fails, i.e., in general $D_X Y \neq D_Y X$ and $\nabla_X Y \neq \nabla_Y X$.

An example of this:

Let $e_1, e_2$ be the standard basis of $\mathbb{R}^2$ with coordinates $(x^1, x^2)$. Then one has $D_{e_i} e_j = 0$ for all $i, j$. Choosing $X := x^1 \cdot e_2$, $Y := e_1$, we get

$$D_X Y = D_{x^1 e_2} e_1 = x^1 D_{e_2} e_1 = 0,$$
but \( D_Y X = D_{e_1}(x^1 e_2) = x^1 D_{e_1}e_2 + D_{e_1}x^1 e_2 = e_2 \neq 0 \).

4.5. Definition. For two vector fields \( X, Y \) in \( \mathbb{R}^{n+1} \) or two vector fields along \( f \), the expression

\[
[X, Y] := D_X Y - D_Y X
\]

is called the Lie bracket of \( X \) and \( Y \). One has \( [X, Y] = \nabla_X Y - \nabla_Y X \), if \( X \) and \( Y \) are tangent vector fields. Moreover,

\[
\left[ \frac{\partial f}{\partial u^i}, \frac{\partial f}{\partial u^j} \right] = 0 \quad \text{because} \quad \frac{\partial^2 f}{\partial u^i \partial u^j} = \frac{\partial^2 f}{\partial u^j \partial u^i}.
\]

In addition, in arbitrary coordinates, one has

\[
[X, Y] = \sum_{i,j} \left( \xi^i \frac{\partial \eta^j}{\partial u^i} - \eta^j \frac{\partial \xi^i}{\partial u^i} \right) \frac{\partial f}{\partial u^j},
\]

if \( X = \sum_i \xi^i \frac{\partial f}{\partial u^i}, Y = \sum_j \eta^j \frac{\partial f}{\partial u^j} \). An abbreviated notation for this is

\[
[X, Y]^j = X(Y^j) - Y(X^j).
\]

Here the index \( j \) denotes the \( j \)th coordinate.

CONSEQUENCE: For given vector fields \( X, Y \) the vanishing of the Lie bracket is a necessary condition for \( X \) and \( Y \) to be basis vector fields \( X = \frac{\partial f}{\partial u^i}, Y = \frac{\partial f}{\partial u^j} \).

4.6. Theorem. The covariant derivative \( \nabla \) depends only on the first fundamental form, and as such is a quantity of the intrinsic geometry of the surface.

PROOF: We set \( X = \sum_i \xi^i \frac{\partial f}{\partial u^i} \) and \( Y = \sum_j \eta^j \frac{\partial f}{\partial u^j} \). In order to determine \( \nabla_X Y \), it is sufficient to know the quantities \( \langle \nabla_X Y, \frac{\partial f}{\partial u^k} \rangle \) for all \( k \). From the calculus rules 4.4 we get the equation

\[
\nabla_X Y = \sum_i \xi^i \nabla_{\frac{\partial f}{\partial u^n}} Y = \sum_i \xi^i \sum_j \nabla_{\frac{\partial f}{\partial u^n}} \left( \eta^j \frac{\partial f}{\partial u^j} \right) =
\]

\[
= \sum_{i,j} \xi^i \left( \frac{\partial \eta^j}{\partial u^i} \frac{\partial f}{\partial u^j} + \eta^j \nabla_{\frac{\partial f}{\partial u^n}} \frac{\partial f}{\partial u^j} \right),
\]
and consequently
\[ \langle \nabla_X Y, \frac{\partial f}{\partial u^k} \rangle = \sum_{ij} \xi^i \left( \frac{\partial g_{ij}}{\partial u^k} \frac{\partial f}{\partial u^k} + \eta^j \left( \nabla_{\frac{\partial g_{ij}}{\partial u^k}} \frac{\partial f}{\partial u^k} \right) \right). \]

Here we use the notation
\[ \Gamma_{ij,k} := \langle \nabla_{\frac{\partial g_{ij}}{\partial u^k}} \frac{\partial f}{\partial u^k}, \frac{\partial f}{\partial u^k} \rangle, \]
and these quantities are symmetric in the indices \( i \) and \( j \), as we know by 4.5 that the Lie brackets of basis fields vanish. On the other hand, we also have
\[ \frac{\partial}{\partial u^k} g_{ij} = \frac{\partial}{\partial u^k} \left( \frac{\partial f}{\partial u^i}, \frac{\partial f}{\partial u^j} \right) = \Gamma_{ik,j} + \Gamma_{jk,i}. \]

By cyclically permuting the indices one gets
\[ \frac{\partial}{\partial u^i} g_{jk} = \Gamma_{ji,k} + \Gamma_{ki,j}, \quad \frac{\partial}{\partial u^j} g_{ki} = \Gamma_{kj,i} + \Gamma_{ij,k}. \]

From this we get, by adding or subtracting these equations,
\[ 2\Gamma_{ij,k} = -\frac{\partial}{\partial u^k} g_{ij} + \frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki}, \]
which is an expression which clearly depends only on the first fundamental form.

\[ \square \]

4.7. Definition. (Christoffel symbols)

(i) The quantities \( \Gamma_{ij,k} \) defined by the expressions
\[ \Gamma_{ij,k} := I \left( \nabla_{\frac{\partial g_{ij}}{\partial u^k}} \frac{\partial f}{\partial u^k}, \frac{\partial f}{\partial u^k} \right) \]
are called the Christoffel symbols of the first kind.

(ii) The quantities \( \Gamma^k_{ij} \) defined by
\[ \nabla_{\frac{\partial g_{ij}}{\partial u^k}} \frac{\partial f}{\partial u^k} = \sum_k \Gamma^k_{ij} \frac{\partial f}{\partial u^k} \]
are called the Christoffel symbols of the second kind.

(iii) By definition one has \( \Gamma_{ij,k} = \Gamma_{ji,k}, \Gamma^k_{ij} = \Gamma^k_{ji} \) as well as \( \Gamma_{ij,k} = \sum_m \Gamma^m_{ij} g_{mk} \).

Consequence: The first fundamental form \( (g_{ij}) \) uniquely determines the Christoffel symbols and thus also the covariant derivative.
of $X = \sum_i \xi^i \frac{\partial f}{\partial u^i}$ and $Y = \sum_j \eta^j \frac{\partial f}{\partial u^j}$ through the equation

$$\nabla_X Y = \sum_{i,k} \xi^i \left( \frac{\partial \eta^k}{\partial u^i} + \sum_j \eta^j \Gamma^k_{ij} \right) \frac{\partial f}{\partial u^k}.$$  

**4.8. Corollary. (Equations of Gauss and Weingarten)**

For every (hyper-)surface element $f$, the following equations hold:

(i) The **Gauss formula**

$$\frac{\partial^2 f}{\partial u^i \partial u^j} = \sum_k \Gamma^k_{ij} \cdot \frac{\partial f}{\partial u^k} + h_{ij} \cdot \nu.$$  

(ii) The **Weingarten equation**

$$\frac{\partial \nu}{\partial u^i} = -\sum_{j,k} h_{ij} g^{jk} \cdot \frac{\partial f}{\partial u^k} = -\sum_k h^k_i \cdot \frac{\partial f}{\partial u^k}.$$  

The proof is more or less contained in the above definitions (for the Gauss formula) and in 3.9 (for the Weingarten equation). In fact, compare the equations

$$D_X Y = \nabla_X Y + II(X,Y) \nu$$  

and

$$D\nu = -L \circ Df.$$  

Like the Frenet equations in the theory of curves, 4.8 can also be written as a motion of the **Gaussian three-frame** $\frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial u^2}, \nu$ using the following matrix:

$$\frac{\partial}{\partial u^i} \left( \begin{array}{c} \frac{\partial f}{\partial u^1} \\ \frac{\partial f}{\partial u^2} \\ \nu \end{array} \right) = \left( \begin{array}{ccc} \Gamma^1_1 & \Gamma^1_2 & h_{i1} \\ \Gamma^2_1 & \Gamma^2_2 & h_{i2} \\ -h^1_i & -h^2_i & 0 \end{array} \right) \left( \begin{array}{c} \frac{\partial f}{\partial u^1} \\ \frac{\partial f}{\partial u^2} \\ \nu \end{array} \right).$$  

and similarly in higher dimensions.

**4B Parallel displacement and geodesics**

That a vector field $Y$ in Euclidean space is constant means just that the directional derivatives $D_X Y$ vanish in all directions $X$. Since one has to think of the different vectors in space as being based at different points, a constant vector field is characterized by the property that all these vectors are **parallel** to one another (and have the same length). However, the naive parallel transport of $(p, X)$ to $(q, X)$ would not