Problem 1.
Let $R$ be a commutative ring. Let $A \neq 0$ be an ideal in $R[x]$ such that the lowest degree of a nonzero polynomial in $A$ is $n \geq 1$ and such that $A$ contains a monic polynomial of degree $n$. Prove that $A$ is a principal ideal in $R[x]$.

Solution.
The argument is essentially the same as for proving that for a field $F$ all ideals in $F[x]$ are principal.
Let $I$ be an ideal in $R[x]$ such that the lowest degree of a nonzero polynomial in $I$ is $n \geq 1$ and such that $R$ contains a monic polynomial $f(x)$ of degree $n$. We claim that $(f) = I$. Since $f \in I$, it is clear that $(f) \subseteq I$.
Let $g \in I$ be arbitrary. Since $f$ is monic and its leading coefficient is a unit, there exist $q, r \in R[x]$ such that $g = qf + r$ and such that either $\deg(r) < \deg(f)$ or $r = 0$. We claim that $r = 0$. If not then $r \neq 0$ satisfies $\deg(r) < \deg(f) = n$. We have

$$r = g - qf \in I, \quad \text{since } g, f \in I \triangleleft R[x],$$

Thus $r$ is a nonzero polynomial in $I$ of degree $\deg(r) < \deg(f) = n$. This contradicts the choice of $f$. Therefore $r = 0$ so that $q = qf \in (f)$. Since $g \in I$ was arbitrary, it follows that $I \subseteq (f)$. Therefore $I = (f)$, as required.

Problem 2.
Let $f(x) = x^3 + 1 \in \mathbb{Z}_3[x]$. Find the order of the quotient ring $\mathbb{Z}_3[x]/(f)$.

Solution.
Let $g(x) \in \mathbb{Z}_3[x]$ be arbitrary. Then by the division algorithm we know that there exist $q(x), r(x) \in \mathbb{Z}_3[x]$ such that $g(x) = q(x)f(x) + r(x)$ and $\deg r < \deg f = 3$. Note that in this case $q(x)f(x) \in (f)$ and hence $g(x) + (f(x)) = r(x) + (f(x))$ in the quotient ring $\mathbb{Z}_3[x]/(f(x))$.
Since $g(x) \in \mathbb{Z}_3[x]$ was arbitrary, this implies that

$$\mathbb{Z}_3[x]/(f(x)) = \{g(x) + (f(x)) : g(x) \in \mathbb{Z}_3[x]\} = \{r(x) + (f(x)) : r(x) \in \mathbb{Z}_3[x], \deg r \leq 1\}.$$

We now claim that if $r_1, r_2 \in \mathbb{Z}_3[x]$ are such that $\deg(r_1), \deg(r_2) \leq 1$ and $r_1 + (f) = r_2 + (f)$ then $r_1 = r_2$ in $\mathbb{Z}_3[x]$.
Indeed, suppose $r_1, r_2$ are as above, that is $\deg(r_1), \deg(r_2) \leq 1$ and $r_1 + (f) = r_2 + (f)$. Then $r_1 - r_2 \in (f)$ so that $r_1 - r_2 = fh$ for some $h \in \mathbb{Z}_3[x]$. If $h \neq 0$ then $\deg(fh) \geq \deg(f) = 2 > 1$. However $\deg(r_1 - r_2) \leq 1$. Hence $h = 0$ and therefore $r_1 - r_2 = 0$ and $r_1 = r_2$, as claimed.
Thus the order of the quotient ring $\mathbb{Z}_3[x]/(f(x))$ is equal to the number of $r(x) \in \mathbb{Z}_3[x]$ such that $\deg(r) \leq 1$. Every such $r$ has the form $r = a + bx$ where $a, b \in \mathbb{Z}_3$. The number of choices for the pair $a \in \mathbb{Z}_3, b \in \mathbb{Z}_3$ is $3 \cdot 3 = 9$. Hence the quotient ring $\mathbb{Z}_3[x]/(f(x))$ has order 9.