Quiz 6 (Solutions); Friday, March 11, 2005

For each of the following maps determine if it is a homomorphism. If it is not, explain why not.
(a) \( f : \mathbb{R}^2 \to \mathbb{R} \) defined as \( f(x, y) = x + 2y \) for every \( (x, y) \in \mathbb{R}^2 \). Here both \( \mathbb{R}^2 \) and \( \mathbb{R} \) are considered as additive groups.

(b) \( f : S_n \to S_n \) where \( n \geq 3 \) and \( f(\alpha) = (12)\alpha \) for every \( \alpha \in S_n \).

(c) \( f : GL(2, \mathbb{R}) \to \mathbb{R}^\times \) defined as \( f(A) = [\det(A)]^2 \) for every \( A \in GL(2, \mathbb{R}) \). Here the group operation on \( \mathbb{R}^\times \) is the standard multiplication of real numbers.

(d) \( f : GL(2, \mathbb{R}) \to GL(2, \mathbb{R}) \) defined as \( f(A) = A^2 \) for every \( A \in GL(2, \mathbb{R}) \).

Solution.
(a) Yes, this is a homomorphism.
Indeed,
\[
f((x_1, y_1) + (x_2, y_2)) = f(x_1 + x_2, y_1 + y_2) = x_1 + x_2 + 2(y_1 + y_2) = (x_1 + 2y_1) + (x_2 + 2y_2) = f(x_1, y_1) + f(x_2, y_2)
\]
for any \( x_1, x_2, y_1, y_2 \in \mathbb{R} \).

(b) No, this is not a homomorphism since \( f(1) = (12) \neq 1 \).

(c) Yes, this is a homomorphism.

(d) No, this is not a homomorphism.
Indeed,

\[
f \text{ is a homomorphism } \iff f(AB) = f(A)f(B) \text{ for every } A, B \in GL(2, \mathbb{R}) \iff ABAB = A^2B^2 \text{ for every } A, B \in GL(2, \mathbb{R}) \iff BAB = AB^2 \text{ for every } A, B \in GL(2, \mathbb{R}) \iff BA = AB \text{ for every } A, B \in GL(2, \mathbb{R}).
\]

The last statement does not hold since \( GL(2, \mathbb{R}) \) is not abelian.
Note that the function \( f \) does satisfy the properties \( f(I_2) = I_2 \), where \( I_2 \) is the \( 2 \times 2 \) identity matrix, and \( f(A^{-1}) = [f(A)]^{-1} \) for every \( A \in GL(2, \mathbb{R}) \).