Quiz 10 (solutions); Friday, April 22, 2005

For each of the following determine whether it is a field. If not, explain why not.

1. \( \mathbb{Z} \);
2. \( \{ a + bi | a, b \in \mathbb{Q} \} \);
3. \( \mathbb{C}[x] \).

Solution.

(1) \( \mathbb{Z} \) is not a field since \( 2 \in \mathbb{Z} \) has no multiplicative inverse in \( \mathbb{Z} \).

(2) \( S = \{ a + bi | a, b \in \mathbb{Q} \} \) is a field since it is a subfield of \( \mathbb{C} \). Note that if \( z = a + bi \in S, z \neq 0 \) then

\[
\frac{1}{z} = \frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \in S,
\]

since for \( a, b \in \mathbb{Q} \) with \( a^2 + b^2 \neq 0 \) we have \( \frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \in \mathbb{Q} \).

(3) \( \mathbb{C}[x] \) is not a field since \( x \in \mathbb{C}[x] \) has no multiplicative inverse in \( \mathbb{C}[x] \). Indeed, if \( f(x) \in \mathbb{C}[x], f \neq 0 \), then

\[
\deg(xf) = \deg(x) + \deg(f) = 1 + \deg(f) \geq 1 > 0,
\]

and hence \( xf \neq 1 \in \mathbb{C}[x] \).