Homework 9 (selected solutions)

2.104

Prove that no pair of the following groups of order 8 are isomorphic (I am omitting the group $D_8$ since it was not discussed in class): 

$I_8, I_4 \times I_2, I_2 \times I_2 \times I_2, Q$.

Solution.

The group $Q$ is non-abelian while the groups $I_8, I_4 \times I_2, I_2 \times I_2 \times I_2$ are abelian. Therefore $Q$ is not isomorphic to any of the groups $I_8, I_4 \times I_2, I_2 \times I_2 \times I_2$.

The group $I_8$ has an element of order 8 while the groups $I_4 \times I_2, I_2 \times I_2 \times I_2$ do not have elements of order 8. Therefore $I_8$ is not isomorphic to either of the groups $I_4 \times I_2, I_2 \times I_2 \times I_2$.

2.112 (i) How many permutations in $S_5$ commute with $(1\ 2)(3\ 4)$ and how many even permutations commute with $(1\ 2)(3\ 4)$?

(ii) How many permutations in $S_7$ commute with $(1\ 2)(3\ 4\ 5)$?

(iii) Exhibit all the permutations in $S_7$ that commute with $(1\ 2)(3\ 4\ 5)$.

Solution.

First we need to prove:

**Lemma** Let $H \leq G$ be a subgroup of index 2. Let $A \leq G$. Then $[A : A \cap H] \leq 2$.

**Proof.** If $A \subseteq H$ then $A = A \cap H$ and $[A : A \cap H] = 1$. Suppose now that $A \not\subseteq H$. Let $a \in A - H$. Then $G = H \cup aH$. We claim that $A = (A \cap H) \cup a(A \cap H)$. This would imply that $[A : A \cap H] = 2$.

Let $b \in A$ be arbitrary. If $b \in H$ then $b \in A \cap H$. Suppose now $b \not\in H$ then $b \in aH$ and $b = ah$ for some $h \in H$. Therefore $h = a^{-1}b \in A$ since $a, b \in A$. This $h \in A \cap H$ and $b \in a(A \cap H)$. Since $b \in H$ was arbitrary, we have proved that $A = (A \cap H) \cup a(A \cap H)$, as required. \[\square\]

(i) The set of permutations commuting with $x = (1\ 2)(3\ 4)$ in $S_5$ is the centralizer $C_{S_5}(x)$. Therefore $[S_5 : C_{S_5}(x)] = |x^{S_5}|$. The conjugacy class $x^{S_5}$ consists of all permutations in $S_5$ with the cycle structures of being the product of two disjoint cycles. Therefore 

$$|x^{S_5}| = \binom{5}{2} \binom{3}{2} \binom{1}{2} = 15.$$ 

Therefore 

$$|C_{S_5}(x)| = \frac{|S_5|}{[S_5 : C_{S_5}(x)]} = \frac{5!}{15} = 8.$$ 

By the Lemma above $[C_{S_5}(x) : C_{S_5}(x) \cap A_5] \leq 2$ since $[S_5 : A_5] = 2$. The permutation $(1\ 2)$ is odd but it commutes with $x$. Thus $(1\ 2) \in C_{S_5}(x) - A_5$.
and therefore \( |C_{S_5}(x) : C_{S_5}(x) \cap A_5| \neq 1 \). Hence \( |C_{S_5}(x) : C_{S_5}(x) \cap A_5| = 2 \) and so
\[
|C_{S_5}(x) \cap A_5| = \frac{1}{2} |C_{S_5}(x)| = 8/2 = 4.
\]
Thus there are 4 even permutations in \( S_5 \) that commute with \( x \).

(ii) Denote \( y = (1 \ 2)(3 \ 4 \ 5) \in S_7 \). By the same argument as above
\[
|S_7 : C_{S_7}(y)| = |y^{S_7}|.
\]
The conjugacy class \( y^{S_7} \) consists of all permutations in \( S_7 \) with the cycle structure of the disjoint product of a 2-cycle and a 3-cycle. The number of such permutations is:
\[
|y^{S_7}| = \binom{7}{2} 5 \cdot 4 \cdot 3 \cdot \frac{1}{3} = 420.
\]
Therefore the number of elements of \( S_7 \) that commute with \( y \) is
\[
|C_{S_7}(y)| = \frac{|S_7|}{|S_7 : C_{S_7}(y)|} = \frac{7!}{420} = 12.
\]
(iii) We know that for any \( \alpha \in S_7 \)
\[
\alpha y \alpha^{-1} = (\alpha(1) \ \alpha(2)) (\alpha(3) \ \alpha(4) \ \alpha(5)).
\]
Thus \( \alpha y \alpha^{-1} = y \) if and only if \( (\alpha(1) \ \alpha(2)) = (1 \ 2) \) and \( (\alpha(3) \ \alpha(4) \ \alpha(5)) = (3 \ 4 \ 5) \). For each such \( \alpha \) either \( \alpha(6) = 6, \alpha(7) = 7 \) or \( \alpha(6) = 7, \alpha(7) = 6 \).
Thus \( \alpha \) commutes with \( y \) if and only if
\[
(\alpha(1), \alpha(2)) \in \{(1, 2), (2, 1)\} \quad \text{and} \quad (\alpha(3), \alpha(4), \alpha(5)) \in \{(3, 4, 5), (4, 5, 3), (5, 3, 4)\} \quad \text{and} \quad (\alpha(6), \alpha(7)) \in \{(6, 7), (7, 6)\}.
\]
Hence
\[
C_{S_7}(y) = \left\{ \begin{array}{c}
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7)
\end{array} \right\}.
\]