2.85

Prove that every group $U(\mathbb{I}_9) \cong \mathbb{I}_6$ and that $U(\mathbb{I}_{15}) \cong \mathbb{I}_4 \times \mathbb{I}_2$.

Solution.

By Lemma 2.106 $U(\mathbb{I}_9) = \{[1]_9, [2]_9, [4]_9, [5]_9, [7]_9, [8]_9\}$. Thus $|U(\mathbb{I}_9)| = 6$. To prove that $U(\mathbb{I}_9) \cong \mathbb{I}_6$ it suffices to show that $U(\mathbb{I}_9)$ is a cyclic group of order 6. To see that $U(\mathbb{I}_9)$ is a cyclic group of order 6 it is enough to find an element of order 6 in $U(\mathbb{I}_9)$. Take $x = [2]_9$. Then $x^2 = [4]_9$, $x^3 = [8]_9$, $x^4 = [16]_9 = [7]_9$, $x^5 = [32]_9 = [5]_9$, $x^6 = [64]_9 = [1]_9$. Thus $x$ has order 6 in $U(\mathbb{I}_9)$ and therefore $U(\mathbb{I}_9) \cong \mathbb{I}_6$, as required.

By Lemma 2.106


To show that $U(\mathbb{I}_{15}) \cong \mathbb{I}_4 \times \mathbb{I}_2$ it suffices to find subgroups $H, K \triangleleft U(\mathbb{I}_{15})$ such that $H$ is cyclic of order 4, $K$ is cyclic of order 2, such that $HK = U(\mathbb{I}_{15})$ and $H \cap K = \{1\}$.

Take $H = \langle [2]_{15} \rangle \leq U(\mathbb{I}_{15})$ and $K = \langle [14]_{15} \rangle \leq U(\mathbb{I}_{15})$. Note that $[14]_{15} = [-1]_{15}$ and therefore $[14]^2_{15} = [1]^2_{15} = [1]_{15}$, so that $[14]_{15}$ has order 2 in $U(\mathbb{I}_{15})$ and $K = \{[1]_{15}, [14]_{15}\}$ is cyclic of order 2.

Similarly we check that $H = \langle [2]_{15} \rangle = \{[1]_{15}, [2]_{15}, [4]_{15}, [8]_{15}\}$ is cyclic of order 4. Thus $H \cap K = \{[1]_{15}\}$. Since $U(\mathbb{I}_{15})$ is abelian, both $H$ and $K$ are normal on $U(\mathbb{I}_{15})$. We have $[7]_{15} = [-8]_{15} = [8]_{15}[-1]_{15} \in HK$, $[11]_{15} = [-4]_{15} = [4]_{15}[-1]_{15} \in HK$ and $[13]_{15} = [-2]_{15} = [2]_{15}[-1]_{15} \in HK$. Thus $HK = U(\mathbb{I}_{15})$. All the required conditions have been verified and hence $U(\mathbb{I}_{15}) \cong \mathbb{I}_4 \times \mathbb{I}_2$.

2.93

(i) Prove that $Q/Z(Q) \cong V$

(ii) Prove that $Q$ has no subgroup isomorphic to $V$.

Solution.

(i) Recall that

$Q = \{\pm 1, \pm i, \pm j, \pm k\}$

where $i^2 = j^2 = k^2 = -1$ and $ij = k, ji = -k, ki = j, ik = -j, jk = i, kj = -i$.

It follows that elements 1, -1 are the only central elements of $V$ so that $Z(V) = \{1, -1\}$. Hence $|Q/Z(Q)| = 8/2 = 4$. Put $H = Z(Q) = \{\pm 1\}$.

We see that

$Q/H = \{H, iH, kH, jH\}$

and that in $Q/H$

$(iH)^2 = (kH)^2 = (jH)^2 = H, iHjH = jHiH = kH,

kHiH = iHkH = jH, jHkH = kHjH = iH.$
Thus $Q/H$ is a group of order 4 where every nontrivial element has order 2 and where the product of any two distinct nontrivial elements is equal to the remaining nontrivial element. Therefore $Q/H \cong V$, as required.

(ii) From the definition of $Q$ we see that the elements $\pm i, \pm k, \pm j$ have order 4 and the only element of order 2 in $Q$ is $-1$. However the group $V$ has three elements of order 2. Therefore $Q$ does not have a subgroup isomorphic to $V$. 