2.60
(i) Prove that every group $G$ with $|G| < 6$ is abelian.
(ii) Find two nonisomorphic groups of order 6.

Solution.
(i) If $|G| < 6$ then $|G| \in \{1, 2, 3, 4, 5\}$.
First we show that every group $G$, where $|G| = p$ is a prime, is abelian.
Indeed, let $|G| = p > 1$ be a prime. Take any $g \in G$ such that $g \neq 1$. Then ord$(g) > 1$. We know that ord$(g)$ divides $|G| = p$. Since $p$ is a prime, it follows that ord$(g) = p$. Hence $p = ord(g) = |\langle g \rangle| = |G|$. Therefore $G = \langle g \rangle$, so that $G$ is cyclic and therefore abelian.
This shows that if $|G| \in \{2, 3, 5\}$ then $G$ is abelian. If $|G| = 1$ then $G$ is the trivial group and hence it is abelian. It remains to consider the case when $|G| = 4$.
Suppose first that $G$ has an element $g$ of order 4. Then $G = \langle g \rangle$ is cyclic of order 4 and therefore $G$ is abelian.
Suppose now that $G$ has no elements of order 4, so that every element of $G$ has order smaller than 4. The order of every element of $G$ must be a divisor of $|G| = 4$. Hence every nontrivial element of $G$ has order 2. This means that for every $g \in G$ we have $g^2 = 1$. By the result of problem 2.38 it follows that $G$ is abelian.
(ii) The groups $S_3$ and $C_6$ both have order 6. The group $C_6$ is abelian while $S_3$ is not. Therefore $C_6$ is not isomorphic to $S_3$.

2.64
(i) Find a subgroup $H \leq S_4$ such that $H \cong V$ but $H \neq V$.
(ii) Show that $H$ is not normal in $S_4$.

Solution.
(i) The group $V$ has the properties that $|V| = 4$, that $x^2 = 1$ for every $x \in V$ and that the product of any two distinct nontrivial elements of $V$ is equal to the remaining nontrivial element. Moreover, these properties completely determine the multiplication table for $V$ and hence any group with these properties is isomorphic to $V$.
Consider $H = \{1, (1 2), (3 4), (1 2)(3 4)\} \leq S_4$. It is easy to see that $H \leq S_4$. Moreover, $|H| = 4$, the square of every element of $H$ is trivial and the product of any two distinct nontrivial elements of $H$ is equal to the remaining nontrivial element. Therefore $H \cong V$.
Any bijection between $H$ and $V$ which takes $1 \in H$ to $1 \in V$ is an isomorphism between $H$ and $V$.
(ii) We have $(1 2) \in H$ but
$$(1 2 3 4)(1 2)(1 2 3 4)^{-1} = (1 3) \notin H.$$ Therefore $H$ is not normal in $G$. 