Math 416, Extra credit set 1. Due Friday, October 21

PRINT YOUR NAME:

Note: To receive any points for the extra credit problems, you must provide extremely careful and detailed write-up of the problems you are solving. Your solutions must be clear, complete and correct both in terms of math and in terms of exposition.

Let $V$ be a vector space over $\mathbb{R}$. Let $r \geq 1$ be an integer. An $r$-tensor on $V$ is a function $T : V \times \cdots \times V \to \mathbb{R}$ which is multi-linear, that is, such that for every $1 \leq i \leq r$, for all $v^1, \ldots, v^r, v'^i \in V$ and for every $\alpha, \beta \in \mathbb{R}$ we have:

$$T(v^1, \ldots, \alpha v^i + \beta v'^i, \ldots, v^r) = \alpha T(v^1, \ldots, v^i, \ldots, v^r) + \beta T(v^1, \ldots, v'^i, \ldots, v^r).$$

An $r$-tensor $T$ on $V$ is called symmetric if for every $v^1, \ldots, v^r \in V$ and every permutation $j_1, \ldots, j_r$ of the numbers $1, \ldots, r$ we have

$$T(v^1, \ldots, v^r) = T(v^{j_1}, \ldots, v^{j_r}).$$

An $r$-tensor $T$ on $V$ is called alternating if for every $v^1, \ldots, v^r \in V$ and all $1 \leq i < j \leq r$ we have

$$T(v^1, \ldots, \hat{v}^i, \ldots, \hat{v}^j, \ldots, v^r) = -T(v^1, \ldots, \hat{v}^j, \ldots, \hat{v}^i, \ldots, v^r).$$

Denote the space of all $r$-tensors on $V$ by $T_r(V)$, denote the space of all symmetric $r$-tensors on $V$ by $S_r(V)$ and denote the space of all alternating $r$-tensors on $V$ by $A_r(V)$.

Also, for $r = 0$ put $T_0(V) = S_r(V) = A_r(V) = \mathbb{R}$.

Thus the spaces $T_r(V)$, $S_r(V)$ and $A_r(V)$ are vector spaces over $\mathbb{R}$ with respect to the obvious operations of pointwise addition and multiplication by a scalar of tensors.

**Problem 1.**

(1) Suppose that $\dim(V) = n \geq 1$. For every $r \geq 0$ find the dimension $\dim(T_r(V))$.

(2) Suppose that $\dim(V) = n \geq 1$. For every $r \geq 0$ find the dimension $\dim(A_r(V))$.

(3) Suppose that $\dim(V) = n \geq 1$. Find the dimensions $\dim(S_3(V))$ and $\dim(S_4(V))$.

Let $j_1, \ldots, j_r$ be a permutation of the numbers $1, \ldots, r$. We put $\delta^{1,\ldots,r}_{j_1,\ldots,j_r} = 1$ if it takes an even number of swaps (where a swap consists in interchanging two numbers in a list) to get the list $1, \ldots, r$ from the list $j_1, \ldots, j_r$. We put $\delta^{1,\ldots,r}_{j_1,\ldots,j_r} = -1$ if it takes an odd number of swaps to get the list $1, \ldots, r$ from the list $j_1, \ldots, j_r$. [You can take it for granted, that $\delta^{1,\ldots,r}_{j_1,\ldots,j_r}$ is well-defined,
that is, that the parity of the number of swaps necessary to get the list $1,\ldots,r$ from the list $j_1,\ldots,j_r$ depends only on the list $j_1,\ldots,j_r$ and not on the particular sequence of swaps].

Let $r,s\geq 0$, and let $V$ be a vector space over $\mathbb{R}$. Let $\eta \in \mathcal{A}_r(V)$ and $\omega \in \mathcal{A}_s(V)$.

Define a function $$\eta \wedge \omega : V \times \cdots \times V \to \mathbb{R}$$ as follows: For any $v_1,\ldots,v_{r+s} \in V$

$$\eta \wedge \omega(v_1,\ldots,v_{r+s}) := \sum_{J=(j_1,\ldots,j_{r+s})} \delta_{j_1,\ldots,j_{r+s}} \eta(v_{j_1},\ldots,v_{j_r})\omega(v_{j_{r+1}},\ldots,v_{j_{r+s}}).$$

Here the summation is taken over all permutations $J = (j_1,\ldots,j_{r+s})$ of the list $1,\ldots,r+s$.

**Problem 2.**

Let $V$ be a vector space over $\mathbb{R}$. Let $\eta \in \mathcal{A}_r(V)$ and $\omega \in \mathcal{A}_s(V)$, where $r,s \geq 0$.

(1) Prove that $\eta \wedge \omega \in \mathcal{A}_{r+s}(V)$.

(2) Prove that $\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega$.

(3) Prove that if $\eta \in \mathcal{A}_r(V)$, $\omega \in \mathcal{A}_s(V)$ and $\tau \in \mathcal{A}_t(V)$ then $\eta \wedge (\omega \wedge \tau) = (\eta \wedge \omega) \wedge \tau$. 