Problem 1. [20 points] In this problem you need to mark each option as either TRUE or FALSE. No explanations of your answers are necessary.

(a) [5 points] Let \( f(t) \) be a \( 2\pi \)-periodic function such that \( f(t) = t^3 + 2t \), where \(-\pi < x < \pi\). Let

\[
f(t) \sim \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nt + \sum_{n=1}^\infty b_n \sin nt
\]

(1) [1 point] We have \( b_n = 0 \) for all \( n \geq 1 \). [FALSE]
(2) [2 points] There is some \( n \geq 0 \) such that \( a_n \neq 0 \). [FALSE]
(3) [2 points] We have \( e^{a_0 - 2a_5} = 1 \). [TRUE]

(b) [6 points]

Consider the equation

\[
y'' + 3y' + 2y = x \cos x + 3x e^{-x}
\]

(1) [2 points] The equation (*) has general solution

\[
y = c_1 e^{-x} + c_2 e^{-2x},
\]

where \( c_1, c_2 \in \mathbb{R} \) are arbitrary constants. [FALSE]

(2) The form of \( y_p \) according to the Method of Undetermined Coefficients is

\[
y_p = (Ax + B) \cos x + (A_1 x + B_1) e^{-x}.
\]

[FALSE]

(3) [2 points]

The form of \( y_p \) according to the Method of Undetermined Coefficients is

\[
y_p = (Ax + B) \cos x + x(A_1 x + B_1) e^{-x}.
\]

[FALSE]

Note: the correct form for \( y_p \) is:

\[
y_p = (Ax + B) \cos x + (A' x + B') \sin x + x(A_1 x + B_1) e^{-x}.
\]
(c)[9 points]
Let \( f(t) \) be a \( 2\pi \)-periodic function such that
\[
   f(t) = \begin{cases} 
   t^2 + 3, & 0 < t < \pi \\
   -t + 1, & -\pi < t < 0 
   \end{cases}
\]
Let \( f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \) be the Fourier Series of \( f(t) \).

1. [4 points] We have
\[
   \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n = 2
\]
[TRUE] Use the Convergence Theorem for \( t = 0 \).

2. [4 points] We have
\[
   \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n - b_n \sin n) = 2
\]
[TRUE] Use the Convergence Theorem for \( t = 1 \).

3. [1 point] The series
\[
   \sum_{n=1}^{\infty} (a_n \cos 9n + b_n \sin 9n)
\]
diverges. [FALSE] The Convergence Theorem for \( t = 9 \) implies that the series converges.

**Problem 2.** [20 points] Consider the following eigenvalue problem.

\[
   (\dagger) \quad y'' + \lambda y = 0, \quad y'(0) = y'(3) = 0.
\]
Find all those eigenvalues that are \( \geq 0 \) and find their corresponding eigenfunctions.

**Solution.**

First, we check if \( \lambda = 0 \) is an eigenvalue. For \( \lambda = 0 \) the main equation \( y'' + \lambda y = 0 \) takes the form \( y'' = 0 \) and its has general solution \( y = Ax + B \) where \( A, B \in \mathbb{R} \) are arbitrary constants.

Then \( y' = A \) and the condition \( y'(0) = y'(3) = 0 \) yields \( A = 0 \) and no restrictions on \( B \). Thus for \( \lambda = 0 \) the system (\( \dagger \)) has the solution \( y = B \) where \( B \in \mathbb{R} \) is arbitrary. Therefore \( \lambda = 0 \) is an eigenvalue with an eigenfunction \( y = 1 \).

Now let \( \lambda > 0 \). There is a unique \( \alpha > 0 \) such that \( \lambda = \alpha^2 \). The main equation becomes \( y'' + \alpha^2 y = 0 \) and its general solution is \( y = A \cos \alpha x + B \sin \alpha x \), where \( A, B \in \mathbb{R} \) are arbitrary constants. Hence \( y' = -A\alpha \sin \alpha x + B\alpha \cos \alpha x \). From \( y'(0) = 0 \) we get \( B\alpha \cdot 1 = 0 \).
and therefore \( B = 0 \). Thus \( y = A \cos \alpha x \) and \( y' = -A \alpha \sin \alpha x \). The condition \( y'(3) = 0 \) gives us

\[-A \alpha \sin 3\alpha = 0\]

If \( \sin 3\alpha \neq 0 \) then \( A = B = 0 \), and therefore \( y = 0 \), so that \( \lambda = \alpha^2 \) is not an eigenvalue.

If \( \sin 3\alpha = 0 \) then \( A \) is arbitrary and (§) has the solution \( y = A \cos \alpha x \) where \( A \in \mathbb{R} \) is an arbitrary constant.

We have \( \sin 3\alpha = 0 \) if and only if \( 3\alpha = \pi n \) for some \( n \in \mathbb{Z} \) (where we must have \( n > 0 \) since by assumption \( \alpha > 0 \)) and hence \( \alpha = \frac{\pi n}{3} \).

This gives us eigenvalues \( \lambda_n = \frac{\pi^2 n^2}{9} \) with corresponding eigenfunctions \( y_n = \cos \frac{\pi nx}{3} \), where \( n = 1, 2, 3, \ldots \).

Combining this information with the case \( \lambda = 0 \) considered earlier, we conclude that the eigenvalue problem (§) has nonnegative eigenvalues \( \lambda_n = \frac{\pi^2 n^2}{9} \) with corresponding eigenfunctions \( y_n = \cos \frac{\pi nx}{3} \), where \( n = 0, 1, 2, 3, \ldots \).

**Problem 3.** [20 points] Let \( f(t) \) be a 2-periodic function such that the Fourier series of \( f(t) \) is

\[ f(t) = \sum_{n \text{ odd}} \frac{1}{n^2 - 2n^4} \sin \pi nt. \]

Find a formal trigonometric series solution of the following problem:

\[
\begin{aligned}
x'' - 5x &= f(t), & 0 < x < 1 \\
x(0) &= x(1) = 0.
\end{aligned}
\]

**Solution.**

We will look for the solution of this system in the form

\[ x = \sum_{n=1}^{\infty} b_n \sin \pi nt. \]

Note that the condition \( x(0) = x(1) = 0 \) is clearly satisfied by any such expression. By termwise differentiation we get

\[
\begin{aligned}
x' &= \sum_{n=1}^{\infty} \pi nb_n \cos \pi nt \\
x'' &= \sum_{n=1}^{\infty} -\pi^2 n^2 b_n \sin \pi nt
\end{aligned}
\]
Substituting these formulas for $x''$ and $x$ into the main equation of the system, we get

$$
\sum_{n=1}^{\infty} -\pi^2 n^2 b_n \sin \pi nt - 5 \sum_{n=1}^{\infty} b_n \sin \pi nt = \sum_{n \text{ odd}} \frac{1}{n^2 - 2n^4} \sin \pi nt
$$

By equating the coefficients at the terms $\cos \pi nt$ on the right and the left hand sides, we get

$$
b_n = \begin{cases} 
\frac{1}{(2n^4 - n^2)(\pi^2 n^2 + 5)}, & n \text{ odd} \\
0, & n \text{ even.} 
\end{cases}
$$

Therefore

$$
x = \sum_{n \text{ odd}} \frac{1}{(2n^4 - n^2)(\pi^2 n^2 + 5)} \sin \pi nt.
$$

**Problem 4.** [20 points]

Find a particular solution of the following equation on the interval $-\infty < x < \infty$:

$$
y'' + 4y' + 4y = xe^{3x}
$$

GIVE ALL THE DETAILS OF YOUR WORK.

**Solution.**

The complimentary homogeneous equation $y'' + 4y' + 4y = 0$ has characteristic equation $r^2 + 4r + 4 = (r + 2)^2 = 0$. Therefore the complimentary solution is

$$
y_c = c_1 e^{-2x} + c_2 xe^{-2x}.
$$

Therefore by the Method of Undetermined Coefficients we can look for a particular solution $y_p$ of the form

$$
y_p = (Ax + B)e^{3x} = Ax e^{3x} + Be^{3x}.
$$

Note that there is no duplication in this case. Differentiating the above formula we get

$$
y'_p = Ae^{3x} + 3Axe^{3x} + 3Be^{3x} = (A + 3B)e^{3x} + 3Axe^{3x}
$$

$$
y''_p = 3(A + 3B)e^{3x} + 3 Ae^{3x} + 9Axe^{3x} = (6A + 9B)e^{3x} + 9Axe^{3x}.
$$
Substituting these formulas for \( y, y', y'' \) into the main equation we get
\[
(6A + 9B)e^{3x} + 9Ax e^{3x} + 4(A + 3B)e^{3x} + 12Ax e^{3x} + 4Be^{3x} = xe^{3x}
\]
\[
(6A + 9B + 4A + 12B + 4B)e^{3x} + (9A + 12A + 4A)xe^{3x} = xe^{3x}
\]
\[
(10A + 25B)e^{3x} + 25Ax e^{3x} = xe^{3x}
\]
From here \( 25A = 1 \), \( 10A + 25B = 0 \) and therefore \( A = \frac{1}{25} \) and
\( B = -\frac{2}{125} \).
Thus
\[
y_p = \frac{1}{25}xe^{3x} - \frac{2}{125}e^{3x}.
\]

**Problem 5.** [20 points]
Find the Fourier Sine Series of the following function
\[ f(t) = 2t, \quad 0 < t < 3. \]

**Solution.**
We have \( L = 3 \) and for \( n \geq 1 \)
\[
b_n = \frac{2}{3} \int_0^3 2t \sin \frac{\pi nt}{3} \, dt = \left[ \text{via } u = 2t, \, dv = \sin \frac{\pi nt}{3} \, dt, \, v = -\frac{3}{\pi n} \cos \frac{\pi nt}{3} \right]
\]
\[
= \left[ -\frac{4t}{\pi n} \cos \frac{\pi nt}{3} \right]_0^3 + \int_0^3 \frac{4t}{\pi n} \cos \frac{\pi nt}{3} \, dt = -\frac{12}{\pi n} \cos \pi n + \frac{12}{\pi n^2} \left[ \sin \frac{\pi nt}{3} \right]_0^3
\]
\[
= -\frac{12}{\pi n} \cos \pi n = \frac{12 \cdot (-1)^{n+1}}{\pi n}.
\]
Therefore the Fourier Sine Series of \( f(t) \) is
\[
\sum_{n=1}^{\infty} \frac{12 \cdot (-1)^{n+1}}{\pi n} \sin \frac{\pi nt}{3}.
\]