Problem 1. [20 points] In this problem you need to mark each option as either TRUE or FALSE. No explanations of your answers are necessary.

(a) [5 points] The differential equation \( y' = 2xy + 1 \)
   (1) [1 point] is separable. [FALSE]
   (2) [1 point] is homogeneous (in the sense of Ch. 1.6). [FALSE]
   (3) [1 point] is linear [TRUE]
   (4) [2 points] has at most one solution on the interval \((-\infty, \infty)\) [FALSE]

(b) [6 points]
   (1) [2 points] The substitution \( v = y/x \) transforms a homogeneous equation into a separable equation. [TRUE]

   (2) [2 points] The substitution \( v = y^{1-n} \) (for an appropriate \( n \)) transforms a Bernoulli equation into a separable equation. [FALSE]

   (3) [2 points] The substitution \( v = ax + by + c \) transforms an equation \( y' = F(ax + by + c) \) (where \( b \neq 0 \)) into a linear first-order equation. [FALSE]

(c) [9 points]
   (1) [3 points] The problem \( y'' + 2y' - 4y = 0, y(0) = 1 \)
   has infinitely many solutions on \((-\infty, \infty)\). [TRUE]

   (2) [3 points] The functions
   \[
   f(x) = |x|, g(x) = \begin{cases} 
   2x, & x \geq 0 \\
   -2x, & x < 0 
   \end{cases}
   \]

   are linearly dependent on \((-\infty, \infty)\). [TRUE]

   (3) [3 points] Whenever \( f(x) \) and \( g(x) \) are functions such that \( W(f, g)(0) = 0 \) then \( f(x) \) and \( g(x) \) are linearly dependent on \((-\infty, \infty)\). [FALSE]

Problem 2. [20 points] Solve the following initial value problem on \( \mathbb{R} \):

\[
y'' + 6y' + 9y = 0, y(0) = 0, y'(0) = 2.
\]
Solution.
We first find the general solution of the equation $y'' + 6y' + 9y = 0$ on $(-\infty, \infty)$. The characteristic equation is:

$$\lambda^2 + 6\lambda + 9 = 0, \quad (\lambda + 3)^2 = 0,$$

and hence the general solution is

$$y = c_1 e^{-3x} + c_2 xe^{-3x}, \quad \text{where } c_1, c_2 \in \mathbb{R}.$$

Differentiating this formula we get

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 xe^{-3x}.$$

Using the conditions $y(0) = 0, y'(0) = 2$ we get:

$$c_1 e^0 + c_2 \cdot 0 \cdot e^0 = 0, \quad -3c_1 e^0 + c_2 e^0 - 3c_2 \cdot 0 \cdot e^0 = 2$$

$$c_1 = 0, \quad -3c_1 + c_2 = 2$$

$$c_1 = 0, \quad c_2 = 2.$$

Hence the solution of the original initial value problem is $y = 2 xe^{-3x}$.

**Problem 3.** [20 points] Solve the following equation

$$y' = \frac{x + y}{x - y}$$

assuming $x > 0$.

**Solution.**

Dividing both the numerator and the denominator in the right-hand side by $x$ we get

$$\frac{dy}{dx} = \frac{1 + y/x}{1 - y/x}.$$

This is a homogeneous equation in the sense of Ch. 1.6. We use the substitution $v = y/x$. Hence

$$y = vx, \frac{dy}{dx} = \frac{dv}{dx} x + v.$$

Thus after the substitution $v = y/x$ the equation becomes

$$\frac{dv}{dx} x + v = \frac{1 + v}{1 - v}$$

$$\frac{dv}{dx} x = \frac{1 + v}{1 - v} - v = \frac{1 + v^2}{1 - v}$$

$$\frac{1 - v}{1 + v^2} dv = \frac{dx}{x}.$$
This is a separable equation. Hence we get

\[
\int \frac{1-v}{1+v^2} dv = \int \frac{dx}{x}
\]

\[
\int \frac{1}{1+v^2} dv - \frac{1}{2} \int \frac{2v}{1+v^2} dv = \int \frac{dx}{x}
\]

arctan \(v\) - \(\frac{1}{2}\) \(\ln(1 + v^2)\) = \(\ln x + C\).

Using \(v = y/x\) we get

\[
\arctan \frac{y}{x} - \frac{1}{2} \ln(1 + \frac{y^2}{x^2}) = \ln x + C,
\]

where \(C \in \mathbb{R}\) is an arbitrary constant.

This is the general solution, given in implicit form, of the original differential equation.

**Problem 4.** [20 points] Verify that the following equation is exact and then solve it:

\[(*)\]

\[
(x^3 + \frac{y}{x}) \, dx + (y^2 + \ln x) \, dy = 0
\]

**Solution.**

We have

\[
\frac{\partial}{\partial y} (x^3 + \frac{y}{x}) = \frac{1}{x}
\]

\[
\frac{\partial}{\partial x} (y^2 + \ln x) = \frac{1}{x}.
\]

Thus \(\frac{\partial}{\partial y}(x^3 + \frac{y}{x}) = \frac{\partial}{\partial x}(y^2 + \ln x)\) and therefore \((*)\) is an exact equation.

We need to find a function \(F(x, y)\) such that \(\frac{\partial F}{\partial x} = x^3 + \frac{y}{x}\) and \(\frac{\partial F}{\partial y} = y^2 + \ln x\). From \(\frac{\partial F}{\partial x} = x^3 + \frac{y}{x}\) we get

\[
F(x, y) = \int (x^3 + \frac{y}{x}) \, dx
\]

\[
F(x, y) = \frac{x^4}{4} + y \ln x + g(y)
\]

Where \(g = g(y)\) is some function depending on \(y\) only. Now using \(\frac{\partial F}{\partial y} = y^2 + \ln x\) we get:
\[
\ln x + g'(y) = y^2 + \ln x
\]
\[
g'(y) = y^2 \quad \text{and hence}
\]
\[
g(y) = \frac{y^3}{3}.
\]
Thus the function \( F(x, y) = \frac{x^4}{4} + y \ln x + \frac{y^3}{3} \) satisfies the requirements that \( \frac{\partial F}{\partial x} = x^3 + \frac{y}{x} \) and \( \frac{\partial F}{\partial y} = y^2 + \ln x \).

Therefore the general solution of (*) is given implicitly by
\[
\frac{x^4}{4} + y \ln x + \frac{y^3}{3} = C,
\]
where \( C \in \mathbb{R} \) is an arbitrary constant.

**Problem 5.** [20 points]

Solve the following equation on the interval \( x > 0 \):

\[
2xy' - 5y = 6x^3y^4.
\]

**Solution.** Dividing the equation by \( 2x \) we get

\[
y' - \frac{5}{2x}y = 3x^2y^4.
\]

This is a Bernoulli equation with \( n = 4 \). We use the substitution \( v = y^{1-4} = y^{-3} \). Hence \( y = v^{-1/3} \) and \( \frac{dv}{dx} = -\frac{1}{3}v^{-4/3}\frac{dv}{dx} \). Substituting this in the above equation we get

\[
-\frac{1}{3}v^{-4/3}\frac{dv}{dx} - \frac{5}{2x}v^{-1/3} = 3x^2v^{-4/3}
\]

multiply the equation by \(-3v^{4/3}\) to get:

\[
\frac{dv}{dx} + \frac{15}{2x}v = -9x^2
\]

This is a linear 1-st order equation. We then compute the integrating factor:

\[
\rho(x) = e^{\int \frac{15}{2x} dx} = e^{\frac{15}{2} \ln x} = x^{15/2}.
\]

Multiplying the last equation by \( x^{15/2} \) we get:
\[ x^{15/2} \frac{dv}{dx} + \frac{15}{2} x^{13/2} = -9x^{19/2} \]
\[ \frac{d}{dx}(x^{15/2}v) = 9x^{19/2} \]
\[ x^{15/2}v = - \int 9x^{19/2} \, dx \]
\[ x^{15/2}v = -9 \cdot \frac{2}{21} x^{21/2} + C = -\frac{6}{7} x^{21/2} + C \]
\[ v = -\frac{6}{7} x^3 + Cx^{-15/2} \]

Now using \( y = v^{-1/3} \) we conclude that the general solution of the original equation on the interval \( x > 0 \) is
\[ y = \left( -\frac{6}{7} x^3 + Cx^{-15/2} \right)^{-1/3} \]
where \( C \in \mathbb{R} \) is an arbitrary constant.