Additional practice problems about equivalence relations

Note that some of these problems are pretty hard, so you should not necessarily expect them to have simple solutions.

Problem 0.
(1) Give an example of a relation on \(\mathbb{N}\) which is reflexive and transitive but not symmetric.
(2) Give an example of a relation on \(\mathbb{N}\) which is transitive and symmetric but not reflexive.
(3) Give an example of a relation on \(\mathbb{N}\) which is reflexive and symmetric but not transitive.
(4) Give an example of a relation on \(\mathbb{N}\) which is anti-symmetric and transitive but not reflexive.

Problem 1.
(1) Give an example of an equivalence relation \(R\) on \(\mathbb{N}\) such that for every \(n \in \mathbb{N}\) the equivalence class \([n]\) is finite and has size 2.
(2) Give an example of an equivalence relation \(R\) on \(\mathbb{N}\) such that \(|\mathbb{N}/R| = 2\) and such that there exist \(x, y \in \mathbb{R}\) such that the set \([x]\) is finite but the set \([y]\) is infinite.
(3) Give an example of an equivalence relation \(R\) on \(\mathbb{R}\) such that for every \(x \in \mathbb{R}\) we have \(|[x]| = |\mathbb{N}|\) and such that the quotient set \(\mathbb{R}/R\) is infinite.
(4) Give an example of an equivalence relation \(R\) on \(\mathbb{R}\) such that for every \(x \in \mathbb{R}\) we have \(|[x]| = |\mathbb{R}|\) and such that \(|\mathbb{R}/R| = |\mathbb{N}|\).

Problem 2.
Consider the relation \(R\) on \(\mathbb{R}^2\), described in part (9) of Example 1.3 from the handout on the equivalence relations.
(1) Prove that \(R\) is an equivalence relation on \(\mathbb{R}^2\).
(2) For a point \(p \in \mathbb{R}^2\), describe geometrically the equivalence class \([p]_R\). [Hint: the answer will depend on whether or not \(p\) is equal to the origin \((0, 0)\).]
(3) Prove that \(T = \{(x, 0)|x \geq 0\}\) is a transversal for \(R\).
(4) Explain why \(\{(x, 0)|x \in \mathbb{R}\}\) is not a transversal for \(R\).
(5) Prove that \(|\mathbb{R}^2/R| = |\mathbb{R}|\).
(6) Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be defined as \(f(x, y) = e^{x^2+y^2}\). Determine whether or not \(f\) factors through to a function \(\mathbb{R}^2/R \to \mathbb{R}\).
(7) Let \(g : \mathbb{R}^2 \to \mathbb{R}\) be defined as \(g(x, y) = x + y\). Determine whether or not \(g\) factors through to a function \(\mathbb{R}^2/R \to \mathbb{R}\).
Problem 3.
Let \( f : X \to Y \) be a function. Define a relation \( R \) on \( X \) as the set of all pairs \((x, x') \in X \times X\) such that \( f(x) = f(x') \).
(a) Prove that \( R \) is an equivalence relation on \( X \).
(b) Prove that \( f \) factors through to a function \( \overline{f} : X/R \to Y \) and that \( \overline{f} \) is injective.
(c) Prove that \(|X/R| = |f(X)|\).

Problem 4.
Let \( X \) be a set and let \( S \subseteq X \times X \) be a relation on \( X \).
(a) Prove that there exists an equivalence relation \( R \subseteq X \times X \) on \( X \) such that \( S \subseteq R \). Hint: One of the relations given in Example 1.3 in the handout on the equivalence relation can be used as \( R \) here.
(b) Let \( S' \) be the intersection of all the equivalence relations on \( X \) which contain \( S \). Prove that \( S' \) is an equivalence relation on \( X \). This \( S' \) is called the equivalence relation on \( X \) generated by \( S \).
(c) Let \( X = \{1, 2, 3, 4\} \) and \( S = \{(1, 2), (2, 4)\} \). Compute the equivalence \( S' \) on \( X \) generated by \( S \).

Problem 5.
Taking for granted that \(|R| = |R^2|\), prove that there exists an equivalence relation \( S \) on \( X \) such that \(|R/S| = |R|\) and such for every \( x \in R \) we have \(|[x]_S| = |R|\).
Hint: First construct an equivalence relation \( R \) on \( \mathbb{R}^2 \) such that \(|\mathbb{R}^2/R| = |\mathbb{R}|\) and such for every \((a, b) \in \mathbb{R}^2\) we have \(|[(a, b)]_S| = |\mathbb{R}|\). Then use a bijection between \( \mathbb{R}^2 \) and \( \mathbb{R} \) to “transport” \( S \) to an equivalence relation \( R \) on \( \mathbb{R} \).

Problem 6.
Let \( R \) be an equivalence relation on a set \( X \) and let \(*\) be a binary operation on \( X \), that is a function \( * : X \times X \to X \).
(1) Suppose that for any \( x, y, x', y' \in X \) such that \( xRx' \) and \( yRy' \) we have \((x * y)R(x' * y')\). For all \( a, b \in X \) put
\[
[a]_R * [b]_R := [a * b]_R.
\]
Prove that \( * \) is a well-defined binary operation on \( X/R \).
(2) Give an example of a set \( X \), a binary operation \( * \) on \( X \) and an equivalence relation \( R \) on \( X \) such that the formula (1) above DOES NOT give a well-defined binary operation on \( X/R \).
Hint: Try to use \( X = \mathbb{N} \) and \( * \) being the standard multiplication of natural numbers, and choose a specific equivalence relation \( R \) on \( \mathbb{N} \) for which (1) fails.

Problem 7.
Recall that a linear order on a set $X$ is a binary relation $R$ on $X$ such that $R$ is reflexive, transitive and anti-symmetric and such that for every $x, y \in X$ at least one of $xRy$, $yRx$ holds.

Let $F$ be a field and let $P \subseteq F$ be a positive set in $F$ in the sense of Definition 1.40 on p. 16 in the book.

Let $R = \{(a, b) \in F \times F|b - a \in P \text{ or } b = a\}$.

Prove that $R$ is a linear order on $F$.

**Problem 8.**

Let $X$ be a nonempty set and let $(X_j)_{j \in J}$ be a family of nonempty subsets $X_j \subseteq X$ of $X$ such that $\cup_{j \in J} X_j = X$ and such that whenever $j, j' \in J$ and $j \neq j'$ then $X_j \cap X_{j'} = \emptyset$.

Define a relation $R$ on $X$ as

$R = \{(x, y) \in X \times X| \text{there exists } j \in J \text{ with } x \in X_j, \text{ and } y \in X_j\}$.

(1) Prove that $R$ is an equivalence relation on $X$.

(2) Prove that if $x \in X$ and $j \in J$ are such that $x \in X_j$ then $[x]_R = X_j$.

**Problem 9.** Let $X = \{(m, n) \in \mathbb{Z} \times \mathbb{Z}|n \neq 0\}$.

Let $R$ be the relation on $X$ consisting of all pairs $((m, n), (m', n')) \in X \times X$ such that $mn' = m'n$.

(1) Prove that $R$ is an equivalence relation on $X$.

(2) Prove that the function $f : X \to \mathbb{Q}$ given by $f(m, n) = \frac{m}{n}$, factors through to a function $\overline{f} : X/R \to \mathbb{Q}$.

(3) Prove that the function $\overline{f}$, constructed in (2), is a bijection.

**Problem 10.** [This is a pretty hard problem meant for those students who are familiar with some linear algebra].

Let

$B = \{(v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2|(v_1, v_2) \text{ is a basis of } \mathbb{R}^2\}$.

Let $R$ be a relation on $B$ consisting of all pairs $((v_1, v_2), (v_1', v_2')) \in B \times B$ such that $\det \begin{bmatrix}a & c \\ b & d \end{bmatrix} > 0$, where $a, b, c, d \in \mathbb{R}$ are such that $v_1' = av_1 + bv_2$ and $v_2' = cv_1 + dv_2$.

(1) Show that $R$ is an equivalence relation on $B$.

(2) Show that $|B/R| = 2$.

(3) Show that $T = \{((1, 0), (0, 1)), ((1, 0), (0, -1))\}$ is a transversal for $R$. 