Additional practice problems about countability and cardinality

Note that some of these problems are pretty hard, so you should not necessarily expect them to have simple solutions.

Problem 0. Let \( S = \{a + bi | a, b \in \mathbb{Z}\} \subseteq \mathbb{C} \).
Prove that \( S \) is countable.

Problem 1/2. Let \( A, B, C \) be sets such that \( |A| < |B| \) and \( B \subseteq C \).
Prove that \( |A| < |C| \).

Problem 1. Prove that if \( A \) and \( B \) are sets such that \( |A| = |B| \) then \( |P(A)| = |P(B)| \). Hint: Assuming that \( f : A \rightarrow B \) is a bijection, construct a bijection \( g : P(A) \rightarrow P(B) \).

Problem 2. Prove that \( |\mathbb{R}| = |A| \), where \( A = \mathbb{Q} \cup (3, \infty) \). Hint: Use part (6) of Theorem 1.4 from the handout on cardinality and countability.

Problem 3.
(a) Construct an explicit bijection between \([0, 1)\) and \((0, 1)\). Hint: Solve Problem 6 below first.
(b) Construct an explicit bijection between \([0, 1)\) and \(\mathbb{R}\).

Problem 4. Show that the set \( \mathbb{Z}[x] \) of all polynomials with integer coefficients is countable.
Hint. Prove first that for every integer \( n \geq 1 \) the set \( \mathbb{P}_n \) of all of all polynomials of degree \( \leq n \) with integer coefficients is countable. Then use the fact that the union of countably many countable sets is countable.

Problem 5. A number \( \alpha \in \mathbb{R} \) is called an algebraic integer if there exists a monic polynomial \( p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x] \) of degree \( n \geq 1 \) such that \( p(\alpha) = 0 \).
Prove that the set \( W \) of all algebraic integers is countable.
Hint: Use the fact that the set of monic polynomials with integer coefficients is countable (this can be deduced from the result of Problem 4) and that every such polynomial has only finitely many roots in \( \mathbb{R} \).

Problem 6. Prove that if \( X \) is an infinite set and \( x_0 \in X \) then \( |X| = |X \setminus \{x_0\}| \).

Problem 7.
For a set \( X \) denote by \( F(X, \mathbb{Z}) \) the set of all functions from \( X \) to \( \mathbb{Z} \).
Prove that for every set \( X \) we have \( |X| < |F(X, \mathbb{Z})| \).
Hint. Construct an injective function \( P(X) \rightarrow F(X, \mathbb{Z}) \) and use the fact that \( |X| < |P(X)| \).

Problem 8.
Prove that the set of all circles in $\mathbb{R}^2$ with center $p = (x, y)$ and radius $r$, such that $r > 0$ is a positive rational number and such that $x, y \in \mathbb{Z}$, is countable.

**Problem 9.**

For each number $x \in (0, 1)$ fix a decimal representation of $x = 0.a_1(x)a_2(x)\ldots a_n(x)\ldots$ such that $a_i(x) \in \{0, 1, 2, \ldots, 9\}$ and such that this decimal expression does not terminate in an infinite string of the digit 9.

(a) Consider the function $f : (0, 1) \times (0, 1) \to (0, 1)$ given by the formula

$$f(x, y) = 0.a_1(x)a_1(y)a_2(x)a_2(y)a_3(x)a_3(y)\ldots$$

where $x, y \in (0, 1)$.

Prove that $f$ is injective.

(b) Use the result of (a) to conclude that $|(0, 1)| = |(0, 1) \times (0, 1)|$.

**Problem 10.**

Prove that if $A$ and $B$ are sets such that $A \cap B = \emptyset$ and $|A| = |\mathbb{R}|$ and $|B| = |\mathbb{R}|$ then $|A \cup B| = |\mathbb{R}|$.

**Problem 11.**

Prove that if $B$ is a countable set then $|\mathbb{R} \cup B| = |\mathbb{R}|$.

**Hint:** For simplicity assume first that $\mathbb{R} \cap B = \emptyset$ and that $B$ is countably infinite, and construct a bijection $f : \mathbb{R} \cup B \to \mathbb{R}$ such that $f(x) = x$ for every $x \in \mathbb{R} \setminus \mathbb{N}$ and that $f(B \cup \mathbb{N}) = \mathbb{N}$.