Math 347, H/wk 7 (Solutions)
Due Wednesday, March 8

Recall that a set $X$ is called countable if either $X$ is finite or there exists a bijection $f : \mathbb{N} \to X$. In the latter case $X$ is called countably infinite.

You are allowed to use the following results that were stated in class:

1. If $X$ is countable and $Y \subseteq X$ then $Y$ is countable.
2. If $X$ and $Y$ are countable then $X \times Y$ is countable.
3. If $X$ is countable and $f : X \to Y$ is a surjective function then $Y$ is countable.
4. If $J$ is a countable set and for every $j \in J X_j$ is countable then the set $\bigcup_{j \in J} X_j$ is countable.
5. The interval $(0, 1)$ is uncountable.
6. The set $\mathbb{Q}$ is countable.

Problem 1. Prove that if $A$ is an uncountable set and $A \subseteq B$ then $B$ is also uncountable.

Solution.
Suppose, on the contrary, that there exist sets $A, B$ such that $A \subseteq B$, and $A$ is uncountable but $B$ is countable. Since $B$ is countable, statement (1) above implies every subset of $B$ is countable and, in particular $A$ is countable. However, this contradicts our assumption that $A$ is uncountable.

Problem 2.
(a) Show that the set $\mathbb{R}$ is uncountable.
(b) Prove that $|\mathbb{R}| = |A|$, where $A = (-\infty, -1] \cup (3, \infty)$.

Solution.
(a) Since, by statement (5) above, the interval $(0, 1)$ is uncountable and since $(0, 1) \subseteq \mathbb{R}$, the result of Problem 1 now implies that $\mathbb{R}$ is also uncountable.

(b) We will construct injective functions $f : A \to \mathbb{R}$ and $g : \mathbb{R} \to A$. It will then follow by Theorem 4.47 on p. 91 in the book that $|\mathbb{R}| = |A|$.

Define $f : A \to \mathbb{R}$ by the formula $f(x) = x$ for every $x \in A$. It is obvious that $f$ is injective.

Recall that the function $y = \arctan(x)$ maps $\mathbb{R}$ bijectively to the interval $(-\pi/2, \pi/2)$. The function $y = x + 10$ maps the interval $(-\pi/2, \pi/2)$ bijectively to the interval $(10 - \pi/2, 10 + \pi/2)$. Since $10 - \pi/2 \geq 8 > 3$, we have $(10 - \pi/2, 10 + \pi/2) \subseteq (3, \infty) \subseteq A$. Define a function $g : \mathbb{R} \to A$ by the formula $g(x) = 10 + \arctan x$ for all $x \in \mathbb{R}$. Then by construction $g$ is injective.

Thus we have constructed injective functions $f : A \to \mathbb{R}$ and $g : \mathbb{R} \to A$. It now follows from Theorem 4.47 on p. 91 in the book that $|\mathbb{R}| = |A|$.

Problem 3. Show that for every $n \in \mathbb{N}$ if $A_1, \ldots, A_n$ are countable then $A_1 \times \cdots \times A_n$ is countable. [Hint: There is a natural bijection between the sets $A \times B \times C$ and $(A \times B) \times C$.]
Solution.

We will prove by induction on \(n\) that if \(n \in \mathbb{N}\) and \(A_1, \ldots, A_n\) are countable then \(A_1 \times \cdots \times A_n\) is countable.

For \(n = 1\) the statement is obvious. For \(n = 2\), if \(A_1\) and \(A_2\) are countable then, by statement (2) above, the set \(A_1 \times A_2\) is again countable, as required.

Now let \(n \in \mathbb{N}\), \(n \geq 3\) and suppose it is known that the product of any \(n - 1\) countable sets is countable.

Let \(A_1, \ldots, A_n\) be \(n\) countable sets. Put \(A' = A_1 \times \cdots \times A_{n-1}\). By the inductive assumption the set \(A'\) is countable. We construct a function \(f : A' \times A_n \rightarrow A_1 \times \cdots \times A_n\) as follows. For any \(x = (a_1, \ldots, a_{n-1}) \in A'\) and any \(a_n \in A_n\) put \(f(x, a_n) := (a_1, \ldots, a_{n-1}, a_n)\). It is obvious that the function \(f\) is surjective. [In fact, \(f\) is bijective, but we don’t need to use this fact]. Since \(A'\) and \(A_n\) are countable, statement (2) above implies that \(A' \times A_n\) is also countable. Since \(f : A' \times A_n \rightarrow A_1 \times \cdots \times A_n\) is a surjective function, statement (3) above implies that the set \(A_1 \times \cdots \times A_n\) is countable, as required. This completes the inductive step.

Problem 4. Show that the set

\[ F = \{p + q\sqrt{2} | p, q \in \mathbb{Q}\} \]

is countable.

Solution.

We know, by statement (6) above, that \(\mathbb{Q}\) is countable. Therefore, by statement (2) above, the set \(\mathbb{Q} \times \mathbb{Q}\) is also countable.

Now consider the function \(f : \mathbb{Q} \times \mathbb{Q} \rightarrow F\) defined by the formula \(f(p, q) = p + q\sqrt{2}\), where \(p, q \in \mathbb{Q}\). By construction, the function \(f\) is surjective. Since \(\mathbb{Q} \times \mathbb{Q}\) is countable, statement (3) above implies that \(F\) is countable.

Problem 5. Prove that the set of all polynomials of degree \(\leq 3\) with integer coefficients is countable. [Hint: This problem has something to do with Problem 3 above.]

Solution.

Let \(P_3\) be the set of all all polynomials of degree \(\leq 3\) with integer coefficients. Thus

\[ P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 | a_0, a_1, a_2, a_3 \in \mathbb{Z}\}. \]

Since the set \(\mathbb{Z}\) is countable, the result of Problem 3 above implies that \(\mathbb{Z}^4 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\) is also countable.

Define a function \(f : \mathbb{Z}^4 \rightarrow P_3\) by the formula

\[ f(a_0, a_1, a_2, a_3) = a_0 + a_1x + a_2x^2 + a_3x^3 \]

where \(1, a_2, a_3 \in \mathbb{Z}\). By construction the function \(f\) is surjective. Hence statement (3) above implies that \(P_3\) is countable.

Problem 6. Prove that the set \(X\) of all finite nonempty strings over the English alphabet is countable. [Hint: This problem has something to do with statement (4) above.]
Solution.
For every \( n \in \mathbb{N} \) let \( X_n \) be the set of all strings of length \( n \) over the English alphabet. Then for each \( n \in \mathbb{N} \) we have \( |X_n| = 26^n \), so that the set \( X_n \) is finite and thus countable.

We have \( X = \cup_{n \in \mathbb{N}} X_n \). Since \( \mathbb{N} \) is countable and for every \( n \in \mathbb{N} \) the set \( X_n \) is countable, statement (4) now implies that \( X \) is countable.

**Problem 7.** Prove that the set
\[
A = \{ \alpha \in \mathbb{R} \mid \text{there exist } a, b, c, d \in \mathbb{Z} \text{ satisfying } \alpha^4 + a\alpha^3 + b\alpha^2 + c\alpha + d = 0 \}
\]
is countable. [**Hint:** This problem has something to do with statement (4) and Problem 3 above.]

**Solution.**
We know that every polynomial of degree 4 with real coefficients has at most 4 roots in \( \mathbb{R} \).

Since the set \( \mathbb{Z} \) is countable, the result of Problem 3 above implies that \( \mathbb{Z}^4 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) is also countable.

For every \( j = (a, b, c, d) \in \mathbb{Z}^4 \) denote by \( A_j \) the set of all real roots of the polynomial \( x^4 + ax^3 + bx^2 + cx + d \). Then every \( j \in \mathbb{Z}^4 \) we have \( |A_j| \leq 4 \), so that \( A_j \) is finite and in particular countable.

By construction,
\[
A = \cup_{j \in \mathbb{Z}^4} A_j.
\]
Since \( \mathbb{Z}^4 \) is countable and each set \( A_j \) is countable, statement (4) implies that the set \( A \) is countable, as required.