3. INDUCTION

3.1. A sequence of statements where the 100th statement is the first one false. If \( P(n) \) is \( n < 100 \), then \( P(1), \ldots, P(99) \) are true but \( P(100) \) is false.

3.2. Falsity of a sequence of statements. We are given \( P(1), P(2), \ldots \) such that \( P(1) \) is false, and such that whenever \( P(k) \) is false, also \( P(k+1) \) is false. Define \( Q(n) \) by \( Q(n) = \neg P(n) \). The hypotheses imply that \( Q(1) \) is true and that whenever \( Q(k) \) is true, also \( Q(k+1) \) is true. By the principle of induction, all \( Q(n) \) are true, and hence all \( P(n) \) are false.

3.3. Induction in both directions. We are given statements with an integer parameter such that \( P(0) \) is true, and such that whenever \( P(n) \) is true, also both \( P(n+1) \) and \( P(n-1) \) are true. Since \( P(n) \Rightarrow P(n+1) \) or ordinary induction implies that \( P(n) \) is true when \( n \geq 0 \).

Let \( Q(n) = \neg P(n) \). Since \( P(n) \Rightarrow P(n-1) \), ordinary induction implies that \( Q(n) \) is true when \( n \geq 0 \), and hence \( P(n) \) is true when \( n \leq 0 \).

3.4. If \( P(0) \) is true, and the truth of \( P(n) \) implies the truth of \( P(n+1) \) or \( P(n-1) \), then possibly only two of the indexed statements are true. Since \( P(0) \) is true, \( P(1) \) or \( P(-1) \) must be true. However, the truth of \( P(0) \) and \( P(1) \) does not imply that any other statements among those indexed are true, and neither does the truth of \( P(0) \) and \( P(-1) \).

3.5. For \( n \in \mathbb{N} \), \( \sum_{k=1}^{n} (2k+1) = n^2 + 2n \rightarrow TRUE \). For \( n = 1, 2 \cdot 1 + 1 = 3 = 1^2 + 2 \cdot 1 \). If \( \sum_{k=1}^{n} (2k+1) = n^2 + 2n \), then
\[
\sum_{k=1}^{n+1} (2k+1) = (n^2 + 2n) + (2(n+1) + 1) = (n+1)^2 + 2(n+1).
\]

3.6. If \( P(2n) \) is true for all \( n \in \mathbb{N} \), and \( P(n) \Rightarrow P(n+1) \) for all \( n \in \mathbb{N} \), then \( P(n) \) is true for all \( n \in \mathbb{N} \)—FALSE. The statement \( P(1) \) need not be true. For example, suppose that \( P(n) \) is \( n > 1 \). Here \( P(n) \) is true when \( n \) is an even natural number, and \( n > 1 \) implies \( n + 1 > 1 + 1 > 1 \), so this sequence of statements is a counterexample.

3.7. For \( n \in \mathbb{N} \), \( 2n - 8 < n^2 - 8n + 17 \rightarrow FALSE \). The inequality holds when \( n \in \{1, 2, 3, 4\} \), but it fails for \( n = 5 \). In fact, the inequality fails only when \( n = 5 \), since it is equivalent to \( 0 < n^2 - 10n + 25 = (n - 5)^2 \). One can prove that \( 2n - 8 < n^2 - 8n + 17 \) implies \( 2(n+1) - 8 < (n+1)^2 - 8(n+1) + 17 \) when \( n \geq 5 \).

3.8. For \( n \in \mathbb{N} \), \( 2n - 18 < n^2 - 8n + 8 \rightarrow TRUE \). The inequality is equivalent to \( 0 < n^2 - 10n + 26 = (n - 5)^2 + 1 \), which is positive for all \( n \).

Alternatively, one can use induction. Let \( P(n) \) be \( 2n - 18 < n^2 - 8n + 8 \). If \( P(n) \) is true, then \( 2(n+1) - 18 < 2n - 18 + 2 < n^2 - 8n + 10 = (n+1)^2 - 8(n+1) + 8 \). The last expression is less than or equal to \( (n+1)^2 - 8(n+1) + 8 \) when \( -(2n+1) + 10 \leq 0 \), which is true when \( n \geq 9/2 \). We can check explicitly that \( P(1), P(2), P(3), P(4), P(5) \) are true and then use the computation above to complete a proof by induction.

3.9. For \( n \in \mathbb{N} \), \( \frac{2n-18}{n^2-8n+8} < 1 \rightarrow FALSE \). The inequality differs from that in the preceding problem when \( n^2 - 8n + 8 \leq 0 \). It is false for \( n \in \{2, 3, 4, 5, 6\} \).

3.10. For an odd number of odd integers, the sum and the product are odd. We prove this for \( 2n + 1 \) odd integers, where \( n \geq 0 \). For the basis step, one odd integer is an odd integer. The induction step uses the direct computations that the sum of two odd integers is even, while the product of two odd integers is odd. Thus when we add on the last two odd integers to an odd sum, the sum remains odd, and when we multiply on the last two odd integers to an odd product, the product remains odd.

3.11. Every set of \( n \) elements has \( 2^n \) subsets. We use induction on \( n \) to prove this for \( n \geq 0 \). Basis step: The empty set \( \emptyset \) is the only set of 0 elements, and \( \emptyset \) is the only subset of \( \emptyset \), so the formula \( 2^0 \) is correct when \( n = 0 \).

Induction step: Suppose that the claim is true when \( n = k \). Let \( S \) be a set of \( k + 1 \) elements, and let \( x \) be an element of \( S \). The subsets of \( S \) consist of those containing \( x \) and those not containing \( x \). The subsets not containing \( x \) are subsets of \( S \setminus \{x\} \); by the induction hypothesis, there are \( 2^k \) of these. The subsets containing \( x \) consist of \( x \) together with a subset of \( S \setminus \{x\} \); again the induction hypothesis implies that there are \( 2^k \). Thus altogether there are \( 2^k + 2^k = 2^{k+1} \) subsets of \( S \). Since \( S \) was chosen as any set with \( k + 1 \) elements, the claims also holds when \( n = k + 1 \).

3.12. If \( a \in \mathbb{R} \) and \( n \in \mathbb{N} \), then \( \sum_{i=1}^{n} x = nx \). Let \( P(n) \) be \( \sum_{i=1}^{n} x = nx \). We use induction on \( n \). Basis step \( P(1) \) is true: \( x = 1 \cdot x \).

Induction step \( P(k) \Rightarrow P(k + 1) \). The induction hypothesis is \( \sum_{i=1}^{k} x = kx \). Using this and the distributive law yields \( \sum_{i=1}^{k+1} x = kx + x = (k+1)x \).

3.13. The sum and the difference of two polynomials are polynomials. Let \( f \) and \( g \) be polynomials, so that \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{i=0}^{m} b_i x^i \) for some coefficients \( a_0, \ldots, a_m \) and \( b_0, \ldots, b_m \). We may assume that \( n \geq m \) and let \( b_{m+1} = \ldots = b_n = 0 \). Writing \( g(x) = \sum_{i=0}^{m} b_i x^i \) does not change \( g \).
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3.18. If \(0 \leq a_i \leq b_i\) for all \(i \in \mathbb{N}\), then \(\prod_{i=1}^n a_i \leq \prod_{i=1}^n b_i\). We use induction on \(n\). Basis step \((n = 1)\): given by hypothesis.

Induction step \((n > 1)\): The induction hypothesis states that \(\prod_{i=1}^{n-1} a_i \leq \prod_{i=1}^{n-1} b_i\). We use this and Proposition 1.45(F2) (twice, with commutativity of multiplication) to obtain

\[
\prod_{i=1}^n a_i = (\prod_{i=1}^{n-1} a_i) a_n \leq (\prod_{i=1}^{n-1} b_i) a_n \leq (\prod_{i=1}^{n-1} b_i) b_n = \prod_{i=1}^n b_i.
\]

3.19. If \(k \in \mathbb{N}\) and \(x < y < 0\), then \(x^{2k+1} < y^{2k+1}\). Using induction on \(k\), we prove that \(x^{2k+1} < y^{2k+1}\) for each nonnegative integer \(k\). Basis step \((k = 0)\): given by hypothesis.

Induction step \((k > 0)\): We use commutativity and associativity of multiplication. By Proposition 1.46a and \(x < y < 0\), we have \(\neg x > \neg y > 0\). If \(a > b > 0\) and \(c > d > 0\), then two applications of Proposition 1.45(F2) yield \(ac > bc > bd > 0\). With this and Proposition 1.43e, \(x^2 > y^2 > 0\). By the induction hypothesis, \(x^{2k-1} < y^{2k-1} < 0\). By Proposition 1.46a, \(-x^{-2k-1} > -y^{-2k-1} > 0\). Combining this with \(x^2 > y^2\) yields \(-x^{2k+1} > -y^{2k+1} > 0\), by our earlier computation. Now Proposition 1.46a yields \(x^{2k+1} < y^{2k+1} < 0\).

Alternatively, we can verify by induction that the product of an odd number of negative numbers is negative, and that inequalities \(a_i > b_i > 0\) yield \(\prod_{i=1}^n a_i > \prod_{i=1}^n b_i > 0\). Since \(\neg x > \neg y > 0\), this yields \((-x)^{2k+1} > (-y)^{2k+1} > 0\). We transform this to \((-1)^{2k+1} x^{2k+1} > (-1)^{2k+1} y^{2k+1} > 0\). Since \((-1)^{2k+1} < 0\), we obtain \(x^{2k+1} < y^{2k+1} < 0\).

3.20. The proof of Lemma 3.13 in summation notation.

\[
(x-y) \sum_{j=1}^n x^i y^{j-i} = \sum_{j=1}^n x^i y^{j-i} - \sum_{j=1}^n x^{i-1} y^j = \sum_{j=0}^{n-1} x^i y^j - \sum_{j=1}^n x^{i-1} y^j = x^i - y^n.
\]

3.21. The square of a sum. When expanding the product \((\sum_{i=1}^n x_i) (\sum_{i=1}^n x_i)\), each term in the first factor is multiplied by each term in the second factor. Thus \((\sum_{i=1}^n x_i)^2 = \sum_{i,j=1}^n x_i x_j\). After collecting like terms, this can also be written as \((\sum_{i=1}^n x_i)^2 = \sum_{i=1}^n x_i^2 + 2 \sum_{1 \leq i < j \leq n} x_i x_j\).

3.22. For \(a_1, \ldots, a_n \in \mathbb{R}\), \(\sum_{i=1}^n a_i \leq \sum_{i=1}^n |a_i|\). We use induction on \(n\). When \(n = 1\), the two sides are equal. When \(n = 2\), the statement is the ordinary triangle inequality (Proposition 1.3).

For the induction step, suppose that the inequality holds when \(n = k\); this is the induction hypothesis. We prove that if \(k \geq 2\), then the inequality

\[
\sum_{i=1}^{k+1} a_i \leq \sum_{i=1}^{k+1} |a_i|.
\]
3.28. \( \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \). Induction can be used. Alternatively, recognizing that \( \frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1} \) leads to a telescoping sum.

\[
\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left( \frac{1}{i} - \frac{1}{i+1} \right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}
\]

3.29. \( \sum_{i=1}^{n} (2i - 1) = n^2 \).

Proof 1 (using a previous result). \( \sum_{i=1}^{n} (2i - 1) = 2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2n(n+1)/2 - n = n^2 \).

Proof 2 (induction on \( n \)). \( \sum_{i=1}^{n} (2i - 1) = 1^2 \). If \( \sum_{i=1}^{k} (2i - 1) = k^2 \), then \( \sum_{i=1}^{k+1} (2i - 1) = 2k + 1 + \sum_{i=1}^{k} (2i - 1) = 2k + 1 + k^2 = (k + 1)^2 \).

Proof 3 ("counting two ways"). Arrange \( n^2 \) dots in an \( n \) by \( n \) square. We can count these in layers from a corner, starting with 1 in the corner, then 3 around it, then the next 5, and so on. Each successive rim has two more dots than the one before it, so the rim sizes are the first \( n \) odd numbers, which counts all \( n^2 \) dots.

3.30. \( \sum_{i=1}^{n} (2i - 1)^2 = \frac{n(2n-1)(2n+1)}{3} \).

Proof 1 (induction). Basis Step: the formula holds for \( n = 1 \) since \( 2 \cdot 1 - 1 = 1 = 1 \cdot 1 \cdot 3/3 \). Induction Step: we prove that the formula holds when \( n = k + 1 \) under the hypothesis that it holds when \( n = k \). Splitting off the last term of the summation when \( n = k + 1 \) and applying the induction hypothesis to what remains yields

\[
\sum_{i=1}^{n+1} (2i - 1)^2 = (2n + 1)^2 + \sum_{i=1}^{n} (2i - 1)^2 = (2n + 1)^2 + \frac{1}{3} n(2n-1)(2n+1)
\]

\[
= 2n+1 \left[ 3(2n+1) + n(2n-1) \right] = 2n+1 \left[ 3n^2 + 5n + 3 \right] = 2n+1 \left( n+1 \right) \left( 2n+3 \right).
\]

Proof 2 (known formulas). We have proved that \( \sum_{i=1}^{n} i = n(n+1)/2 \) and \( \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6 \). Thus

\[
\sum_{i=1}^{n} (2i - 1)^2 = \sum_{i=1}^{n} \left( 4i^2 - 4i + 1 \right) = 4 \sum_{i=1}^{n} (2i+1)^2 - 4 \sum_{i=1}^{n} i + n
\]

\[
= 2n+1 \left( 3n^2 + 2n + 1 \right) - n(2n+1) = n(2n+1) \left[ \frac{3n^2+2}{3} - 1 \right] = n(2n-1)(2n+1).
\]