1.45. Well-defined functions from \( \mathbb{R} \) to \( \mathbb{R} \).

a) \( f(x) = |x| - 1 \) if \( x < 4 \) and \( f(x) = |x| - 1 \) if \( x > 2 \)—TRUE. When \( 2 < x < 4 \), both \( x \) and \( x - 1 \) are positive, and thus \( |x| - 1 = x - 1 = |x| - 1 \) in the interval of overlap.

b) \( f(x) = |x| - 1 \) if \( x < 2 \) and \( f(x) = |x - 1| \) if \( x > -1 \)—FALSE. When \( 0 < x < 1 \), we have \( |x - 1| = -(x - 1) = -x + 1 \), but \( |x| - 1 = x - 1 \). In this interval the definitions conflict.

c) \( f(x) = (x + 3)^2 - 9/x \) if \( x \neq 0 \) and \( f(x) = 6 \) if \( x = 0 \)—TRUE. When \( x \neq 0 \), there is no division by 0, so the formula for \( f(x) \) yields a real number. There is no overlap between the sets with \( x \neq 0 \) and \( x = 0 \), so each real number has been assigned a unique real number, and \( f \) is well-defined.

d) \( f(x) = ((x + 3)^2 - 9)/x \) if \( x > 0 \) and \( f(x) = x + 6 \) if \( x < 7 \)—TRUE. When \( x > 0 \), we have \( ((x + 3)^2 - 9)/x = x + 6 \).

e) \( f(x) = \sqrt{x^2} \) if \( x \in \mathbb{Z} \) and \( f(x) = x \) if \( x < 1 \)—FALSE. The notation \( \sqrt{x^2} \) denotes the positive square root; thus \( \sqrt{x^2} = -x \) when \( x \) is a negative integer. Thus the function is not well-defined. Furthermore, the function has not been defined at all at real numbers at least 1 that are not integers.

1.46. Images of functions. Let \( S \) denote the image of \( f \). In each case, we specify \( T \) and show that \( S = T \).

a) \( f: \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = x^2/(1 + x^2) \). Let \( T = \{ y \in \mathbb{R} : 0 \leq y < 1 \} \).

In the formula defining the function, the numerator is always nonnegative and the denominator is always positive, so the image is nonnegative. Also the numerator is always less than the denominator, so the image is always less than 1. Thus \( S \subseteq T \).

For each \( y \in T \), we seek \( x \in \mathbb{R} \) such that \( y = f(x) \). Solving for \( x \) shows that when \( x \) is \( \pm \sqrt{y(1 - y)} \), the image is \( y \). Note that the square root is defined when \( y \in T \), because \( 0 \leq y < 1 \) yields \( y(1 - y) \geq 0 \). Thus \( S \subseteq T \).

b) \( f: \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = x/(1 + |x|) \). Let \( T = (-1, 1) \).

In the defining formula, the absolute value of the numerator is always less than the absolute value of the denominator, so \( S \subseteq T \).

For \( y \in T \), we know that the sign of \( x \) must be the same as the sign of \( y \) if \( y = f(x) \). For \( 0 \leq y < 1 \), we solve \( y = x/(1 + x) \) to obtain \( x = y/(1 - y) \). For \( -1 < y \leq 0 \), we solve \( y = x/(1 - x) \) to obtain \( x = y/(1 + y) \). The resulting \( x \) has the right sign, so we have proved \( T \subseteq S \).

1.47. The image of the function \( f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) defined by \( f(a, b) = (a + 1)(a + 2b)/2 \) is the set of all natural numbers that are not powers of 2. We check first that this defines a function from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{N} \). We need that \( (a + 1)(a + 2b)/2 \) is a natural number when \( a, b \in \mathbb{N} \). Since we only add, multiply and divide positive numbers, the result is positive. It is an integer because \( a + 2b \) has opposite parity from \( a + 1 \). With one odd and one even, the product is divisible by 2.

Now we determine the image. Since exactly one of \( a + 1 \) and \( a + 2b \) is odd, and it exceeds 1, we know that \( f(a, b) \) is the product of two positive integers, one of which is odd and exceeds 1. Thus the image does not contain any power of 2.

We must also show that all other natural numbers are in the image. Let \( s \) be an odd factor of \( n \) greater than 1.

When \( s > \sqrt{2n} \), we desire \( a + 2b = s \) and \( (a + 1)/2 = n/s \); the product is \( n \). We set \( a = 2(n/s) - 1 \) and \( b = s^2 - 1 - (2ns/2) \). Since \( s < n, a \) is positive. Since \( s \) and \( a \) are odd, \( b \) is an integer. Since \( s > \sqrt{2n} \), \( b \) is positive. Hence \( n = f(a, b) \) and \( n \) is in the image.

When \( s \leq \sqrt{2n} \), we desire \( a + 1 = s \) and \( (a + 2b)/2 = n/s \); the product is \( n \). We set \( a = s - 1 \) and \( b = n(s) - (a/2) \). Since \( s > 1, a \in \mathbb{N} \). Since \( s \) is even, \( b \) is an integer. Since \( s^2 - 2s \geq (\sqrt{2n})^2 = 2n > b \), \( b \) is positive. Hence again \( n = f(a, b) \) and \( n \) is in the image.

1.48. Descriptions of the function \( f: [0, 1] \rightarrow [0, 1] \) defined by \( f(x) = 1 - x \).

The graph of \( f \) is the line segment in \( \mathbb{R}^2 \) joining \((0, 0)\) and \((1, 0)\). The function can also be described as giving the amount of water left after \( x \) gallons are removed from a full 1-gallon jug. Note that with this description, the domain of the function is the interval \([0, 1]\).

1.49. Properties of functions \( f, g: \mathbb{R} \rightarrow \mathbb{R} \).

a) If \( f \) and \( g \) are bounded, then \( f + g \) is bounded—TRUE. By the definition of bounded function, there exist positive constants \( M_1, M_2 \in \mathbb{R} \) such that, for \( x \in \mathbb{R} \), \( |f(x)| \leq M_1 \) and \( |g(x)| \leq M_2 \). The constant \( M = M_1 + M_2 \) works to show that \( f + g \) is bounded, because applying the triangle inequality yields, for \( x \in \mathbb{R} \),

\[ |(f + g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M_1 + M_2 = M. \]

b) If \( f \) and \( g \) are bounded, then \( fg \) is bounded—TRUE. Using the same approach as in (a), let \( M = M_1, M_2 \). Now

\[ |(fg)(x)| = |f(x)g(x)| = |f(x)||g(x)| \leq M_1 M_2. \]

c) If \( f + g \) is bounded, then \( f \) and \( g \) are bounded—FALSE. The functions \( f, g \) defined by \( f(x) = x \) and \( g(x) = -x \) provide a counterexample. Here \( f \) and \( g \) have unbounded image, but \( f(x) + g(x) = 0 \) for all \( x \).

d) If \( fg \) is bounded, then \( f \) and \( g \) are bounded—FALSE. Define \( f \) by \( f(x) = x \). Define \( g \) by \( g(x) = 1/x \) for \( x \neq 0 \), and \( g(0) = 0 \). In this example, \( f(x) = 1 \) for \( x \in \mathbb{R} \setminus \{0\} \), and \( fg(0) = 0 \). Thus \( fg \) is bounded, but \( f \) and \( g \) are unbounded.

e) If both \( f + g \) and \( fg \) are bounded, then \( f \) and \( g \) are bounded—TRUE. We are given \( M, N \in \mathbb{R} \) such that for all \( x \), \( |f(x) + g(x)| \leq M \) and
\[ |f(x)g(x)| \leq N. \] We show that \( f \) and \( g \) are bounded by showing that \( f^2 \) and \( g^2 \) are bounded. We have
\[
|f(x)^2 + g(x)^2| = |(f(x) + g(x))^2 - 2f(x)g(x)| \\
\leq |(f(x) + g(x))^2| + 2|f(x)g(x)| \\
\leq M^2 + 2N.
\]

Since \( f(x)^2 \) and \( g(x)^2 \) are both nonnegative, we have \( f(x)^2 \) and \( g(x)^2 \) both bounded by \( f(x)^2 + g(x)^2 \). Thus \( |f(x)| \leq \sqrt{M^2 + 2N} \) and \( |g(x)| \leq \sqrt{M^2 + 2N} \).

**1.50.** Images of subsets of the domain of \( f: A \rightarrow B \). \( \text{Note:} \) The original printing incorrectly stated the problem using unions. Part (b) is valid only for intersections. For a subset \( S \) of the domain of \( f \), let \( f(S) = \{ f(x) : x \in S \} \). Let \( C \) and \( D \) be subsets of the domain.

a) \( f(C \cap D) \subseteq f(C) \cap f(D) \). If some \( b \in B \) belongs to \( f(C \cap D) \), then \( f(x) = b \) for some element \( x \) in \( C \cap D \). Since \( x \in C \), \( b \in f(C) \). Since \( x \in D \), \( b \in f(D) \). Thus \( b \in f(C \cap D) \) implies \( b \in f(C) \cap f(D) \).

b) Equality need not hold. Consider \( f: A \rightarrow B \) with \( A = \{ -1, 1 \}, B = \{ 1 \} \), and \( f(-1) = f(1) = 1 \). Let \( C = \{ -1 \} \) and \( D = \{ 1 \} \). Now \( C \cap D \) and hence also \( f(C \cap D) \) is empty, but \( 1 \in f(C) \cap f(D) \).

**1.51.** "Preimage" of subsets of the target of \( f: A \rightarrow B \). For \( S \subseteq B \), let \( f^{-1}(S) = \{ x \in A : f(x) \in S \} \). Let \( X \) and \( Y \) be subsets of \( B \).

a) \( f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y) \). An element of \( A \) has its image in \( X \cup Y \) if and only if its image is in \( X \) or its image is in \( Y \).

b) \( f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y) \). An element of \( A \) has its image in \( X \cap Y \) if and only if its image is in \( X \) and its image is in \( Y \).

**1.52.** For nonnegative \( M, N \), the maximum \( x \) among pairs \( (x, y) \) such that \( |x + y| \leq M \) and \( |xy| \leq N \) is \( x = (M + \sqrt{M^2 + 4N})/2 \). As in Application 1.38, graphing of level sets shows that the maximum occurs when \( x + y = M \) and \( xy = N \). Solving these by \( x(M - x) + N = 0 \) and taking the larger zero yields \( x = (M + \sqrt{M^2 + 4N})/2 \).

**1.53.** Maximization of \( x \) such that \( |x + y| \leq 8 \) and \( |xy| \leq 20 \), using inequalities. We avoid case analysis by squaring the first inequality to get \( x^2 + 2xy + y^2 \leq 64 \). The second inequality implies \( -4xy \leq 20 \). The sum of these is \( (x - y)^2 \leq 144 \), and hence \( |x - y| \leq 12 \).

By the triangle inequality, \( 2|x| \leq |x + y| + |x - y| \leq 8 + 12 = 20 \). Hence \( |x| \leq 10 \). Since \( (x, y) = (10, -2) \) satisfies both inequalities, the answer is 10.

**Comment:** By symmetry, we have the constraints \(-10 \leq x \leq 10 \) and \(-10 \leq y \leq 10 \), but not all pairs \((x, y) \in [-10, 10] \times [-10, 10] \) satisfy the inequalities.

**Chapter 1: Numbers, Sets, and Functions**

1.54. The set \( S = \{(x, y) \in \mathbb{R}^2 : y \leq x \text{ and } x + 3y \geq 8 \text{ and } x \leq 8 \} \).

a) The graph of \( S \) is a triangle with corners \((8, 0), (8, 8), \) and \((2, 2)\). Replacing the inequalities with equalities yields three lines that form the boundary of this triangle. The inequalities restrict the solution points to the side of each line that includes the interior of the triangle.

b) The minimum value of \( x + y \) such that \((x, y) \in S \) is 4. The level sets of \( f(x, y) = x + y \) are lines at an angle of 45 degrees to the horizontal axis. The first one to hit \( S \) is at the point \((2, 2)\).

1.55. If \( F \) is a field consisting of exactly three elements \( 0, 1, x \), then \( x + x = 1 \) and \( x \cdot x = 1 \). We are given that \( x \) is different from both 0 and 1.

If \( y \neq z \), then \( y + x \neq z + x \), since otherwise adding the additive inverse \(-x\) to both sides yields \( y = z \). Thus \( 0 + x, 1 + x, \text{ and } x + x \) are distinct. We have \( 0 + x = x \), and \( 1 + x \) cannot equal 1 since \( x \neq 0 \). Thus \( 1 + x = 0 \), which leaves \( x + x = 1 \). Since nonzero elements have multiplicative inverses, it follows that products of nonzero elements are nonzero; hence \( x \cdot x \neq 0 \). If \( x \cdot x = x \), then multiplication by \( x^{-1} \) yields \( x = 1 \), which is forbidden. Thus \( x \cdot x = 1 \).

\[
\begin{array}{|c|c|} 
\hline 
+ & 0 & 1 & x \\
\hline 
0 & 0 & 1 & x \\
1 & 1 & 0 & x \\
x & x & 0 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|} 
\hline 
\cdot & 0 & 1 & x \\
\hline 
0 & 0 & 0 & 0 \\
1 & 0 & 1 & x \\
x & 0 & 1 & x \\
\hline
\end{array}
\]

1.56. There is a field of size four but none of size six.

Let \( 0, 1, x, y \) be the elements of a field \( F \) with four elements. Multiplying distinct elements by a nonzero element produces distinct elements. Since always \( 0 \cdot z = 0 \) and \( 1 \cdot z = z \), this determines the multiplication table for \( F \). If \( xy = y \) is forbidden by \( x \neq 1 \), and hence we must have \( xy = 1 = yx \).

Similarly, adding an element to distinct elements produces distinct elements, so \( 1 + x \notin \{1, x\} \). If \( 1 + x = 0 \), then \( 0 = x \cdot 0 = x(1 + x) = x + x \cdot x \). This yields \( x \cdot x = 1 \), but we have shown that \( x \cdot x = y \). Thus \( 1 + x = y \). Interchanging \( x \) and \( y \) in this argument yields \( 1 + y = x \). Also, if \( 0 = x + y \), then \( 0 = 0x = (x + y)x = x \cdot x + y \cdot x = y + 1 \), which we have just forbidden.

We have shown that the only possibility for the arithmetic operations in \( F \) is that given below. With this specification of addition and multiplication in \( F \), it is straightforward (but perhaps tedious) to verify that all the field axioms hold.