14. SEQUENCES AND SERIES

14.1. An unbounded sequence that has no convergent subsequence. Let $x_n = n$. The sequence $\langle x_n \rangle$ is unbounded, as are all its subsequences.

14.2. Unbounded increasing sequences satisfying additional conditions.

a) $\lim (a_{n+1} - a_n) = 0$. Let $a_n = \sqrt{n}$. We have $a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = 1/(\sqrt{n+1} + \sqrt{n}) \to 0$.

b) $\lim (a_{n+1} - a_n)$ does not exist. Let $a_n = n^2$. We have $a_{n+1} - a_n = (n+1)^2 - n^2 = 2n + 1$. Thus $\{a_{n+1} - a_n\}$ is unbounded, and the sequence has no limit.

c) $\lim (a_{n+1} - a_n) = L$, where $L > 0$. Let $a_n = nL$. Since $L > 0$, $\langle a \rangle$ is unbounded. We have $a_{n+1} - a_n = (n+1)L - nL = L \to L$.

14.3. Examples of sequences $\langle a \rangle$ and $\langle b \rangle$ such that $\lim a_n = 0$, $\lim b_n$ does not exist, and the specified condition holds.

14.4. If $x_{n+1} = \sqrt{1 + x_n^2}$ for all $n \in \mathbb{N}$, then $\langle x \rangle$ does not converge. If $\langle x \rangle$ converges, with $\lim x_n = L$, then the properties of limits yield $L = \sqrt{1 + L^2}$. This requires $L^2 = 1 + L^2$, which is impossible.

14.5. A counterexample to the following false statement: “If $a_n < b_n$ for all $n$ and $\Sigma b_n$ converges, then $\Sigma a_n$ converges.” Let $b_n = 0$ and $a_n = -1$ for all $n$, then $a_n < b_n$ for all $n$, and $\Sigma b_n = 0$, but $\Sigma a_n$ diverges.

14.6. The expression $0.111\ldots$ is the k-ary expansion of $\frac{1}{k-1}$. The expansion evaluates to the geometric series $\sum_{n=0}^{\infty} (1/k)^n$. This equals $1/k$ times $\sum_{n=0}^{\infty} (1/k)^n$. Since $\sum_{n=0}^{\infty} (1/k)^n = 1/(k-1)$, we obtain $0.111\ldots = \frac{1}{k} - \frac{1}{k^2} = \frac{1}{k-1}$.

14.7. The binary expansions of $\frac{2}{7}$ and $\sqrt{2}$ to six places are $0.0100010$ and $1.0110100$, respectively. We have $(19/64) > (2/7) = (18/63) > (18/64).$ The binary expansion of $18$ is $1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$; thus $18/64 = 0.010001$, since $64 = 2^6$. Since $2/7$ exceeds $18/64$ by less than $1/64$, its expansion agrees with that of $18/64$ through six places.

Using the bisection algorithm produces the same result. $2/7$ is below $1/2$, above $1/4$, below $3/8$, below $5/16$, above $9/32$, below $19/64$. Again the expansion begins $0.1010010$.

For $\sqrt{2}$, we want the largest multiple of $1/2^6$ whose square is less than $2$. The fastest route with a calculator may be to compare squares with $2 \cdot 2^{12} = 8192$. This exceeds $81 \cdot 100 = 90^2$, and $91^2 = 8281.$ Thus we want the binary expansion of $90$, shifted by six places, $90 = 2^6 + 2^4 + 2^3 + 2^1$, so the expansion begins $1.0110100$.

14.8. Let $\langle x \rangle$ be a sequence of real numbers.

a) If $\langle x \rangle$ is unbounded, then $\langle x \rangle$ has no limit—TRUE. The contrapositive of this statement is immediate from the definition of convergence.

b) If $\langle x \rangle$ is not monotone, then $\langle x \rangle$ has no limit—FALSE. The sequence defined by $x_n = (-1)^n/n$ is not monotone, but it converges.

14.9. Properties of convergence. Suppose that $x_n \to L$.

a) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < \varepsilon$—TRUE. Since $x_{n+1} \to L$ and $x_n \to L$, the difference has limit 0, so it is less than $\varepsilon$ for sufficiently large $n$. 

b) There exists \( n \in \mathbb{N} \) such that for all \( \varepsilon > 0 \), \(|x_{n+1} - x_n| < \varepsilon\) — FALSE. The quantifier on \( \varepsilon \) requires that \( x_{n+1} - x_n = 0 \), but there are convergent sequences with no consecutive values equal.

c) There exists \( \varepsilon > 0 \) such that for all \( n \in \mathbb{N} \), \(|x_{n+1} - x_n| < \varepsilon\) — TRUE. As mentioned in part (a), the difference converges to 0. Hence it is a bounded sequence, with some bound \( M \) on \(|x_{n+1} - x_n|\). Choose \( \varepsilon = 2M \).

d) For all \( n \in \mathbb{N} \), there exists \( \varepsilon > 0 \) such that \(|x_{n+1} - x_n| < \varepsilon\) — TRUE.

Let \( y_n = |x_{n+1} - x_n| \). This is now the statement that \( y_n \to 0 \), which was verified in part (a).

14.10. a) If \( \langle x \rangle \) converges, then there exists \( n \in \mathbb{N} \) such that \(|x_{n+1} - x_n| < 1/2^n\) — FALSE. Let \( x_n = \sum_{k=1}^{\infty} (2/3)^k \). This is the sequence of partial sums of a geometric series, converging to \( 1/(1 - 2/3) \), which equals 3. However, \(|x_{n+1} - x_n| = (2/3)^n\), which is larger than \( 1/2^n\).

b) If \(|x_{n+1} - x_n| < 1/2^n \) for all \( n \in \mathbb{N} \), then \( \langle x \rangle \) converges TRUE. We show that \( \langle x \rangle \) is a Cauchy sequence. Given \( \varepsilon > 0 \), choose \( N \) so that \( 1/2^N < \varepsilon/2 \). For \( m > n \geq N \), we have

\[
|x_m - x_n| = \left| \sum_{i=1}^{m-n} (x_{n+i} - x_{n+i-1}) \right| \leq \sum_{i=1}^{m-n} |x_{n+i} - x_{n+i-1}| \leq \sum_{i=1}^{m-n} \frac{1}{2^n} = \frac{1}{2^n} \sum_{i=1}^{m-n} 2^{-1} < \frac{2}{2^n} < \varepsilon
\]

14.11. a) If \( x_1 = 1 \) and \( x_{n+1} = x_n + 1/n \) for \( n \geq 1 \), then \( \langle x \rangle \) is bounded FALSE. For \( n \geq 2 \), we have \( x_n = 1 + \sum_{k=1}^{n-1} \frac{1}{k} \). If \( \langle x \rangle \) is bounded, then \( \sum_{k=1}^{\infty} \frac{1}{k} \) converges, which is false.

b) If \( y_1 = 1 \) and \( y_{n+1} = y_n + 1/n^2 \) for \( n \geq 1 \), then \( \langle y \rangle \) is bounded TRUE. Since \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges to a number \( \alpha \), we have \( 1 \leq y_n = 1 + \sum_{k=1}^{n-1} \frac{1}{k^2} < 1 + \alpha \), and hence \( \langle y \rangle \) is bounded.

14.12. If \( a_n \to 0 \) and \( b_n \to 0 \), then \( \sum a_n b_n \) converges — FALSE. Let \( a_n = \frac{1}{\sqrt{n}} \). We have \( a_n \to 0 \) and \( b_n \to 0 \), but \( \sum a_n b_n = \sum (1/n^2) \). This is the harmonic series, which diverges.

14.13. If \( \langle a \rangle \) converges, then every subsequence of \( \langle a \rangle \) converges and has the same limit as \( a \). Let \( \langle b \rangle \) be a subsequence of \( \langle a \rangle \), with \( b_k = a_{kh} \). For \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies \(|a_n - L| < \varepsilon\), where \( L = \lim a_n \). Let \( K \) be the minimum \( k \) such that \( a_k \geq N \). Now \( k \geq K \) implies \(|b_k - L| = |a_{kh} - L| < \varepsilon\). Thus \( \langle b \rangle \) also satisfies the definition of convergence to \( L \).

14.14. If \( a_n \to L \) and \( b_n \to M \neq 0 \), then \( a_n/b_n \to L/M \). We may assume that \( |b_n| \) has no 0’s, by deleting corresponding terms from both sequences if 0’s occur in \( \langle b \rangle \). In the text we have proved that the limit of the product of two sequences is the product of the limits. Hence it suffices to prove that \( 1/b_n \to 1/M \), because then we can apply the rule for the limit of a product of sequences.

Since \( b_n \to M \neq 0 \), there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies \(|b_n - M| < |M|/2 \) and thus \(|b_n| > |M|/2 \) and \(|1/b_n| < 2/|M| \). Thus the reciprocals of the terms in \( \langle b \rangle \) form a bounded sequence. Let \( M' \) be a bound: always \(|1/b_n| < M' \). Now \( |1/b_n - 1/M'| = |M - b_n|/(|M|b_n) < |M - b_n|/(M'M') \). The sequence \( c_n = |M - b_n|/(M'M') \) is a constant times a sequence converging to 0 (since \( b_n \to M \)), so \( c_n \to 0 \). By Proposition 13.12, we conclude that \( 1/b_n \to 1/M' \).

To prove that \( 1/b_n \to 1/M \) using the definition, we must determine \( N \) for each \( \varepsilon > 0 \) such that \( n \geq N \) implies \(|1/b_n - 1/M| < \varepsilon \). By the convergence of \( \langle b \rangle \), we can make \(|M - b_n| \) as small as desired; we choose \( N_1 \) such that \( n \geq N_1 \) implies \(|b_n - M| < |M|/2 \). This means \(|b_n| > |M|/2 \) and hence \( n \geq N_1 \) implies \(|1/b_n| < 2/|M| \). We can also choose \( N_2 \) such that \( n \geq N_2 \) implies \(|b_n - M| < \varepsilon |M|^2/2 \). Choose \( N = \max(N_1, N_2) \). For \( n \geq N \), we have

\[
|1/b_n - 1/M| = |M - b_n|/(|M|b_n) < (\varepsilon |M|/2)(1/|M|)(2/|M|) = \varepsilon.
\]

Alternatively, one can apply the definition directly to \( a_n/b_n \), using \( a_n - L = a_{kn} - L = a_{kn} - L + L - L \). In this approach, it is still necessary to choose \( N \) large enough to obtain an appropriate bound on \( |1/b_n| \).

14.15. If \( b \leq L + \varepsilon \) for all \( \varepsilon > 0 \), then \( b \leq L \). We prove the contrapositive. If \( b > L \), then let \( \varepsilon = (b - L)/2 \). Since the average of two numbers is between them, we have \( b > (b + L)/2 = L + (b - L)/2 = b + \varepsilon \).

14.16. If \( a_n = p(n)/q(n) \), where \( p \) and \( q \) are polynomials and \( q \) has larger degree than \( p \), then \( a_n \to 0 \). Let \( k, l \) be the degrees of \( p, q \); and let the leading coefficient of \( q \) be \( b \). Let \( g(n) = p(n)/n^l \) and \( h(n) = q(n)/n^l \), so \( a_n = g(n)/h(n) \). The sequence given by \( g(n) \) is a sum of finitely many sequences whose terms have the form \( c/n^j \), where \( c \in \mathbb{R} \) and \( j \in \mathbb{N} \). By the properties of limits, such sequences have limit 0; hence also their sum \( g(n) \to 0 \). The value of \( h(n) \) is \( b \) plus another expression of this form, so \( h(n) \to b \).

\[
\lim a_n = \lim \frac{p(n)}{q(n)} = \lim n^l g(n)/n^l h(n) = \lim g(n)/h(n) = 0/0 = 0.
\]

14.17. If \( a_n = p(n)x^n \), where \( p \) is a polynomial in \( n \) and \( |x| < 1 \), then \( a_n \to 0 \). If \( x = 0 \), then \( a_n = 0 \) and \( a_n \to 0 \). Thus we may assume that \( x \neq 0 \). We prove that \(|a_{n+1}/a_n| \to x \). We have \( a_{n+1}/a_n = xp(n+1)/p(n) \).