Chapter 13: The Real Numbers

13.8. If S is a bounded set of real numbers, and S contains sup(S) and inf(S), then S is a closed interval—FALSE. Counterexamples include the finite set \( S = \{0, 1\} \) and the uncountable set \( S = [0, 1] \cup [2, 3] \).

13.9. If \( f: \mathbb{R} \to \mathbb{R} \) is defined by \( f(x) = \frac{2x - 8}{x^2 + 17} \), then the supremum of the image of \( f \) is 1—TRUE. We show that 1 is an upper bound on \( f(x) \) and that 1 is in the image. The latter claim follows from \( f(5) = 2/2 = 1 \).

Since \( x^2 - 8x + 17 \leq (x - 4)^2 + 1 \), this quadratic polynomial is never zero. Hence the inequality \( f(x) \leq 1 \) is equivalent to \( 2x - 8 \leq x^2 - 8x + 17 \), which is equivalent to \( 0 \leq x^2 - 10x + 25 \). Since \( x^2 - 10x + 25 = (x - 5)^2 \geq 0 \), the desired inequality is always true.

13.10. Every positive irrational number is the limit of a nondecreasing sequence of rational numbers—TRUE. For each irrational number \( \alpha \), let \( a_n \) denote the decimal expansion of \( \alpha \) to \( n \) places. This defines a nondecreasing sequence of rational numbers with limit \( \alpha \).

13.11. a) If \( (a) \) converges and \( \lim a_n < \lim b_n \), then there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies \( a_n < b_n \)—FALSE. Let \( L = \lim a_n \) and \( M = \lim b_n \). Let \( \varepsilon = (M - L)/2 \). The definition of convergence implies that there exist \( N_1 \) and \( N_2 \) such that \( n \geq N_1 \implies |a_n - L| < \varepsilon \) and \( n \geq N_2 \implies |b_n - M| < \varepsilon \). Let \( N = \max\{N_1, N_2\} \). For \( n \geq N \), we have \( a_n < L + \varepsilon < M - \varepsilon < b_n \).

b) If \( (a) \) and \( (b) \) converge and \( \lim a_n \leq \lim b_n \), then there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies \( a_n \leq b_n \)—FALSE. If \( a_n = 2/n \) and \( b_n = 1/n \), then \( \lim a_n = 0 = \lim b_n \), so \( \lim a_n \leq \lim b_n \), but \( a_n > b_n \) for all \( n \).

13.12. If \( S \) is a bounded set of real numbers, and \( x_n \to \sup(S) \) and \( y_n \to \inf (S) \), then \( \lim x_n + y_n \in S \)—FALSE. Consider \( S = (1, 2) \). If \( x_n = 1 \) for all \( n \), and \( y_n = 2 \) for all \( n \), then \( x_n + y_n \) converges to 3, which is not in \( S \).

The counterexample still works when we consider \( \frac{x_n + y_n}{2} \), since \( \frac{x_n + y_n}{2} = \frac{3}{2} \notin S \).

13.13. If \( x > 0 \) and \( x^2 \neq 2 \), then \( y = \frac{1}{2}(x + \sqrt{x}) \) satisfies \( y^2 > 2 \). We show that \( y^2 - 2 \) is a square. We have
\[
y^2 - 2 = \left( \frac{1}{2} x + \frac{\sqrt{x}}{2} \right)^2 - 2 = \frac{1}{4} (x^2 + 4 + \frac{2\sqrt{x}}{x}) - \frac{8}{2} = \frac{1}{4} (x^2 - 4 + \frac{4}{x}) = \frac{1}{4} (x - \frac{2}{x})^2 > 0.
\]
Note that \( x^2 \neq 2 \) implies that \( x - 2/x \neq 0 \).

13.14. To six places, the base 3 expansion of \( 1/10 \) is 0.002200. We have \((73/729) > (1/10) > (72/729) \). The base 3 expansion of 72 is 2200, since \( 72 = 2 \cdot 27 + 2 \cdot 9 + 0 \cdot 3 + 0 \cdot 0 \cdot 1 \). Dividing by 729 = 3² yields 0.002200. Since
1/10 exceeds 72/729 by less than 1/729, the expansion of 1/10 agrees with this through the first six places.

**13.15.** Reciprocals of positive integers with one-digit expansions. In base \( k \), we seek positive integer solutions to \( \frac{1}{n} = \frac{1}{k} \) with \( 1 \leq i < k \). Rewriting this as \( n = ki \), we get a solution for each divisor of \( k \) less than \( k \). For \( k = 10 \), the fractions are 1/2, 1/5, 1/10. For \( k = 9 \), they are 1/3, 1/9, 1/11. For \( k = 8 \), they are 1/2, 1/4, 1/8.

**13.16.** In base 26, the string BAD represents the decimal number 679. \( D_{26}^0 + A_{26}^1 + B_{26}^2 = 3 + 0 + 1(676) = 679 \).

In base 26, the string \( M M M M M M M M M M M \cdots \) represents 12/25. Let \( x \) be the desired value. Note that the value of \( M \) is 12. From \( 26x = M M M M M M M M M M M \cdots \), we have \( 26x = 12 + x \), and thus \( x = 12/25 \).

**13.17.** When \( q \) is odd, the base \( q \) expansion of 1/2 consists of \((q - 1)/2\) in each position. See the more general result in the next solution.

**13.18.** When \( q \equiv 1 \pmod{3} \), the base \( q \) expansion of 1/3 consists of \((q - 1)/3\) in each position. In general, we prove that if \( q \equiv 1 \pmod{k} \), then the base \( q \) expansion of 1/k consists of \((q - 1)/k\) in each position.

The alternative expansion of 1 in base \( q \) consists of \( q - 1 \) in every position. Since \( k(q - 1) \), the distributive law for series allows us to calculate the sum of the series \( \sum (q - 1)x^m \) by dividing each coefficient by an infinite sum

\[
\sum \frac{a_{q-1}}{k} q^{-m}.
\]

**13.19.** If \( f \) is a bounded function on an interval \( I \), then \( \sup((-f(x); x \in I)) = -\inf(f(x); x \in I) \). Let \( \alpha = \sup(-f(x); x \in I) \), and \( S = \{f(x); x \in I\} \). We have \( \alpha \geq -f(x) \) and hence \( -\alpha \leq f(x) \) for all \( x \in I \), so \( -\alpha \) is a lower bound for \( S \).

On the other hand, Prop 13.15 yields a sequence \( (x) \) of numbers in \( I \) such that \( -f(x_n) \to \alpha \). Thus \( f(x_n) \to -\alpha \). We now apply the analogue of Prop 13.15 for infimum. Since \( -\alpha \) is a lower bound for \( S \) and \( -f(x_n) \) defines a sequence of elements of \( S \) converging to \( -\alpha \), we conclude that \( -\alpha = \inf(S) \).

**13.20.** Sequence converging to infimum or to supremum.

a) \( S = \{x \in \mathbb{R}; 0 \leq x < 1\} \). We have \( x_n = 1 - 1/(n + 1) \to 1 = \sup(S) \) and \( y_n = 1/(n + 1) \to 0 = \inf(S) \).

b) \( S = \{\frac{2n+1}{2^k}; n \in \mathbb{N}\} \). The set \( S \) consists of the terms of a sequence that begins 1, 3/2, 1/3, 3/4, .... The constant sequence converges to the supremum: \( x_n = 3/2 = \sup(S) \). A monotone sequence converging to the infimum is given by \( y_n = 3/(2n) \to 0 = \inf(S) \).

**13.21.** The Least Upper Bound Property holds for an ordered field \( F \) if and only if the Greatest Lower Bound Property holds for \( F \). Given a set \( S \), let \( -S \) denote \( \{x : -x \in S\} \). Upper bounds on \( -S \) are the negatives of lower bounds on \( S \), and lower bounds on \( -S \) are the negatives of upper bounds on \( S \). The LUB Property implies for nonempty \( S \) that \( -S \) has at least upper bound \( -\alpha \), which implies that \( S \) has a greatest lower bound \( -\alpha \), and the GLB Property follows. Conversely, the GLB Property implies for nonempty \( S \) that \( -S \) has a greatest lower bound \( -\alpha \), which implies that \( S \) has at least upper bound \( -\alpha \), and the LUB Property follows.

**13.22.** Determination of \( \sup(S) \) and \( \inf(S) \).

a) \( S = \{x : x^2 < 5x\} \). Rewrite \( S \) as \( S = \{x : x(x - 5) < 0\} \). Thus \( x \in S \) if and only if \( x \) and \( x - 5 \) have opposite signs. This requires \( x > 0 \) and \( x < 5 \), and that satisfies, so \( S \) is the open interval \((0, 5) \). This set is bounded by \( 0 \) and \( 5 \), and \( \sup(S) = 5 \) and \( \inf(S) = 0 \).

b) \( S = \{x : 2x^2 < x^3 + x\} \). Rewrite \( S \) as \( S = \{x : x(x - 1)^2 > 0\} \). The condition holds if and only if \( x > 0 \) and \( x \neq 1 \). This set is unbounded, but its infimum is \( 0 \).

c) \( S = \{x : 4x^2 > x^3 + x\} \). The inequality is equivalent to \( x(4x^2 - 4x + 1) < 0 \). The zeros of the quadratic factor are at \( x = 2 \pm \sqrt{3} \). Thus \( S = (-\infty, 0) \cup (2 - \sqrt{3}, 2 + \sqrt{3}) \). The set has no lower bound, but \( \sup(S) = 2 + \sqrt{3} \).

**13.23.** If \( A, B \subset \mathbb{R} \) have upper bounds and \( C = \{x + y : x \in A, y \in B\} \), then \( C \) is bounded and \( \sup(C) \leq \sup(A) + \sup(B) \). Let \( \alpha = \sup(A) \) and \( \beta = \sup(B) \). We prove first that \( \alpha + \beta \) is an upper bound for \( C \). For each \( x \in C \), the definition of \( C \) implies that \( x = z + y \) for some \( z \in A \) and \( y \in B \). By the definition of upper bound, \( x \leq \alpha \) and \( y \leq \beta \). Hence \( z = x + y \leq \alpha + \beta \), and \( \alpha + \beta \) is an upper bound for \( C \).

To prove that \( \alpha + \beta \) is the least upper bound, consider \( q \) such that \( q < \alpha + \beta \). Thus \( q = \alpha + \beta - \varepsilon \) for some \( \varepsilon > 0 \). Since \( \alpha = \sup(A) \), the number \( \alpha - \varepsilon/2 \) is not an upper bound for \( A \), and there exists \( x \in A \) with \( x > \alpha - \varepsilon/2 \). Similarly, there exists \( y \in B \) with \( y > \beta - \varepsilon/2 \). This constructs \( z \in C \) such that \( z = x + y > \alpha + \beta - \varepsilon = q \). Hence \( q \) is not an upper bound for \( C \).

**Alternative proof:** Instead of showing directly that \( C \) has no smaller upper bound, it also suffices to show that \( C \) contains the elements of a sequence converging to \( \alpha + \beta \). This can be obtained by taking a sequence \( (x) \) in \( A \) converging to \( \alpha \) and a sequence \( (y) \) in \( B \) converging to \( \beta \). The sum consists of elements of \( C \) as \( x_n + y_n \to \alpha + \beta \).

**Comment:** Since \( \alpha + \beta \) may not lie in \( C \), one cannot prove that \( \alpha + \beta \) is the least upper bound for \( C \) without using the properties of supremum. For example, if \( A = \{x \in \mathbb{R} : 0 < x < 1\} \) and \( B = \{x \in \mathbb{R} : 2 < x < 3\} \), then \( C = \{x \in \mathbb{R} : 2 < x < 4\} \); none of these sets contains its supremum.
13.24. When \( f, g: \mathbb{R} \to \mathbb{R} \) are bounded functions such that \( f(x) \leq g(x) \) for all \( x \), with images \( F, G \) respectively, the following possibilities may occur (pictures omitted):

a) \( \sup(F) < \inf(G) \). Let \( f(x) = 0 \) and \( g(x) = 1 \) for all \( x \).

b) \( \sup(F) = \inf(G) \). Let \( f(x) = g(x) = 0 \) for all \( x \).

c) \( \sup(F) > \inf(G) \). Let \( f(x) = |x| \) for \( |x| \leq 1 \) and \( f(x) = 1 \) for \( |x| > 1 \).

Let \( g(x) = |x| \) for \( |x| \leq 2 \) and \( g(x) = 2 \) for \( |x| > 2 \). Now \( \sup(f(x)) = 1 \) and \( \inf(g(x)) = 0 \).

13.25. \( \lim \sqrt{1 + \frac{n}{n^2}} = 1 \). Given \( \varepsilon > 0 \), let \( N = \lceil 1/\varepsilon \rceil \). Note that \( \sqrt{1 + \frac{n}{n^2}} < 1 + \frac{1}{n^2} \) when \( n > 0 \). For \( n \geq N \), we have \( \sqrt{1 + \frac{n}{n^2}} - 1 < \left| 1 + \frac{1}{n^2} - 1 \right| = \frac{1}{n^2} < \varepsilon \). Thus \( \sqrt{1 + \frac{n}{n^2}} \to 1 \), by the definition of limit.

Comment: Let \( a_n = \sqrt{1 + \frac{n}{n^2}} \). A less efficient approach first uses MCT to prove that \( \langle a \rangle \) converges. Letting \( L = \lim a_n \), we have \( a_n^2 \to L^2 \). Proving \( a_n^2 \to 1 \) directly yields \( L = \pm 1 \), and positivity of \( a_n \) then yields \( L = 1 \).

13.26. If \( \lim a_n = 1 \), then \( \lim \left[ (1 + a_n)^{-1} \right] = \frac{1}{2} \).

Consider \( \varepsilon > 0 \). Because \( \lim a_n = 1 \), the definition of limit tells us that there exists \( N_1 \in \mathbb{N} \) such that \( n \geq N_1 \) implies \( |a_n - 1| < \varepsilon \). Also \( |1 + a_n| = |1 + 1 + a_n| \leq 2 + |a_n - 1| < 2 + \varepsilon \). Let \( N = N_1 \). Now \( n \geq N \) implies

\[
\left| \frac{1}{1 + a_n} - \frac{1}{2} \right| = \frac{|1 - a_n|}{2(1 + a_n)} = \frac{|a_n - 1|}{2(1 + a_n)} < \frac{\varepsilon}{2(1 + a_n)} < \varepsilon.
\]

Thus \( \left( 1 + a_n \right)^{-1} \to \frac{1}{2} \), by the definition of limit.

13.27. If \( a_n = \sqrt{n^2 + n} - n \), then \( \lim a_n = \frac{1}{2} \). We multiply and divide \( a_n \) by \( \sqrt{n^2 + n} + n \), simplify the result, and use Exercises 13.25–13.26. Thus

\[
a_n = \sqrt{n^2 + n} - n = \frac{(n^2 + n - n)(\sqrt{n^2 + n} + n)}{(\sqrt{n^2 + n} + n)} = \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \frac{n^2 + 1}{\sqrt{1 + 1/n} + 1} - \frac{1}{2}.
\]

13.28. If \( x_n \to 0 \) and \( |y_n| \leq 1 \) for \( n \in \mathbb{N} \), then \( \lim(x_n y_n) = 0 \). One cannot argue that \( \lim(x_n y_n) = \lim(x_n) \lim(y_n) = 0 \cdot 0 = 0 \), since \( \lim(y_n) \) need not exist.

A correct proof uses \( |y_n| \leq 1 \) to argue that \( |x_n y_n| = |x_n| |y_n| \leq |x_n| \). Given \( \varepsilon > 0 \), the convergence of \( \langle x \rangle \) yields \( N \in \mathbb{N} \) such that \( n \geq N \) implies \( |x_n| < \varepsilon \). By our first computation, \( |x_n y_n| \leq |x_n| < \varepsilon \) for such \( n \), and thus \( \lim x_n y_n = 0 \).

13.29. The limit of the sequence \( \langle x_n \rangle \) defined by \( x_n = (1 + n)/(1 + 2n) \) is \( 1/2 \).

Since the denominator exceeds the numerator and both are positive, we have \( 0 < x_n < 1 \) for all \( n \in \mathbb{N} \). We also compute

\[
x_{n+1} - x_n = \frac{n + 2}{2n + 3} - \frac{n + 1}{2n + 1} = \frac{(n + 1)(n + 2) - (2n + 3)(n + 1)}{(2n + 3)(2n + 1)} = \frac{-1}{(2n + 3)(2n + 1)} < 0.
\]

Since \( \langle x \rangle \) is a decreasing sequence bounded below, the Monotone Convergence Theorem implies that \( \lim_{n \to \infty} x_n \) exists.

To prove that \( \lim_{n \to \infty} x_n = 1/2 \), we compute \( x_n - 1/2 = -\frac{1}{2n + 3} - \frac{1}{2} = -\frac{1}{4(n^2 + 1)} \).

Given \( \varepsilon > 0 \), choose \( N \in \mathbb{N} \) such that \( N^2 > 4/\varepsilon \). Now \( n > N \) implies \( |x_n - 1/2| = \frac{1}{4n^2 + 1} < \varepsilon \). Since this holds for each \( \varepsilon > 0 \), we have \( x_n \to 1/2 \), by the definition of limit.

13.30. The sequence \( \langle x \rangle \) defined by \( x_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \) converges. By the Monotone Convergence Theorem, it suffices to prove that \( \langle x \rangle \) is increasing and bounded above by \( 1 \). For the first statement

\[
x_{n+1} - x_n = \sum_{i=1}^{n+1} \frac{1}{n+1+i} - \sum_{i=1}^{n} \frac{1}{n+i} = \frac{1}{2(n+1)} + \frac{1}{2(n+2)} - \frac{1}{2(n+1)} = \frac{1}{2n+2} > 0.
\]

For the second statement, \( x_n = \sum_{i=1}^{n} \frac{1}{i+1} < \sum_{i=1}^{n} \frac{1}{i} = n + 1 \leq 1 \).

13.31. \( x_n = (1 + (1/n)^y \) defines a bounded monotone sequence. Let \( r_n = x_{n+1}/x_n \). We show that \( r_n > 1 \) to prove that \( \langle x \rangle \) is increasing. Writing \( x_n \) as \( \left( \frac{n+1}{n+2} \right)^y \), we have

\[
r_n = \left( \frac{n+2}{n+1} \right)^y \frac{n+2}{n+1} n+2 = \left( \frac{n^2 + 2n}{n^2 + 2n + 1} \right)^y n+1 = \left( 1 - \frac{1}{(n+1)^2} \right)^y n+2 n+1.
\]

Since \( (1 - a^n) \geq 1 - n a \) (Corollary 3.20) when \( a > 0 \), we have

\[
r_n \geq \left( 1 - \frac{n}{(1+2)^2} \right) n+2 n+1 = \frac{n^2 + n + 2}{n^2 + 2n + 1} n+2 = \frac{n^2 + 3n^2 + 3n + 2}{n^2 + 3n^2 + 3n + 1} \geq 1.
\]

To show that \( \langle x \rangle \) is bounded, we write \( x_n = (1 + 1/n)^y = \sum_{k=0}^{n} \binom{n}{k} n^{-k} \).

Since \( \prod_{k=0}^{n} (n-k) < n^k \), we obtain \( x_n \leq \sum_{k=0}^{n} \frac{1}{k!} \). Thus it suffices to show that this sum is bounded. We have \( 1/k! < 1/2^k \) for \( k \geq 4 \). Therefore,

\[
\sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} < \frac{8}{5} + \frac{8}{5} + \frac{8}{5} = \frac{56}{15}.
\]

13.32. The Nested Interval Property. A nested sequence of closed intervals, with \( I_n \) of length \( d_n \), satisfies \( I_{n+1} \subseteq I_n \) for all \( n \) and \( d_n \to 0 \). The Nested