Partial Solutions to H/wk 7

Ch 3.6 no. 1.

Determine which of the following statements are true in neutral geometry.

(a) The Alternate Interior Angle Theorem.

True. This was proved in Theorem 3.4.1.

(b) The converse of the Alternate Interior Angle Theorem.

False. This statement was shown in Theorem 3.4.6 to be equivalent to the Euclidean parallel postulate which need not hold in neutral geometry.

(c) If the Euclidean parallel postulate is true then , when two parallel lines are intersected by a transversal, the pairs of the interior angles on the same side of the transversal are supplementary.

True. The conclusion in the above statement follows immediately from the converse to the Alternate Interior Angle theorem which, as shown in Theorem 3.4.6 follows from the Euclidean parallel postulate.

(d) A Saccheri quadrilateral exists.

True. We can prove the existence of a Saccheri quadrilaterals using only the neutral axioms.

Indeed, by SMSG Postulate 5 there exists a plane and two distinct points A, B in it. The line $\overleftrightarrow{AB}$ separates the plane into two half-planes (Postulate 9). Choose one of these half-planes and then, using the Angle Construction Postulate (Postulate 12) construct two rays $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ contained in that half-plane and perpendicular to the line $\overleftrightarrow{AB}$. Then, using the Ruler Postulate (Postulate 3) construct points C on $\overrightarrow{BB'}$ and D on $\overrightarrow{AA'}$ such that $BC = AB$ and $AD = AB$. Finally, join the points B and C by a line-segment. Then $ABCD$ is a Saccheri quadrilateral: the sides $\overline{AD}$ and $\overline{BC}$ are congruent to each other and perpendicular to the side $\overline{AB}$.

(e) A rectangle exists.

False. This statement is implies (see Theorem 3.6.14) that all triangles have angle sum of 180° which in turn implies (see Theorem 3.6.18) that the Euclidean parallel postulate holds. However, the Euclidean parallel postulate which need not hold in neutral geometry.

(f) If a rectangle exists then every Saccheri quadrilateral is a rectangle.

True. If a rectangle exists then (see Theorem 3.6.15 and Theorem 3.6.18) the Euclidean parallel Postulate holds. This in turn implies by Theorem 3.4.6 that the converse of the Alternate Interior Angle Theorem holds. If $ABCD$ is a Saccheri quadrilateral with base $\overline{AB}$ and legs $\overline{AD}, \overline{BC}$ then the summit $\overline{CD}$ is parallel to the base $\overline{AB}$ (see Corollary 3.6.5). Since $\overline{AD}$ is perpendicular to $\overline{AB}$, the converse of the Alternate interior angle Theorem implies that $\overline{AD}$ is perpendicular to $\overline{CD}$. Similarly, since $\overline{BC}$ is perpendicular to $\overline{AB}$, the converse of the Alternate interior angle Theorem implies that $\overline{BC}$ is perpendicular to $\overline{CD}$. Hence all four angles of $ABCD$ are right and $ABCD$ is a rectangle.

(g) The summit and the base of a Saccheri quadrilateral are parallel.

True by Corollary 3.6.5

(h) If two lines are intersected by a transversal such that the alternate interior angles are congruent, then the lines must have a common perpendicular.

True: see exercise no. 3 in Ch. 3.4

(i) If the Alternate Interior Angle Theorem is true, then its converse is true.

False. The Alternate Interior Angle Theorem is true in neutral geometry as proved in Theorem 3.4.1. However, its converse is shown in Theorem 3.4.6 to be equivalent to the Euclidean parallel postulate that need not hold in neutral geometry.

(j) If there is a triangle with angle sum less than 180° then no rectangle exists.

True. This follows from Theorem 3.6.14.

(k) If no rectangle exists then a triangle having angle sum less than 180° exists.

True. Indeed, suppose not, that is, no rectangle exists but there are no triangles with angle sum less than 180°. By the Saccheri-Legendre Theorem (Theorem 3.5.1) this implies that every triangle has angle sum exactly 180°. Then Theorem 3.6.15 implies that there exists a rectangle, contradicting our assumptions.

(l) If a line is perpendicular to one of two parallel lines then it is perpendicular to the other.

False. By Theorem 3.4.8 this statement is equivalent to the Euclidean parallel postulate which need not hold in neutral geometry.

(m) If a quadrilateral is both Saccheri and Lambert then it is a rectangle.

True. Indeed, in a Saccheri quadrilateral that is also a Lambert quadrilateral one of the summit angles must be right. Since the summit angles of a Saccheri quadrilateral are congruent, the other summit angle must be right also. Thus the quadrilateral has four right angles and so it is a rectangle.

(n) Parallelograms have both pairs of opposite sides congruent.

False. Indeed, suppose the above statement holds. By part (d) there exists a Saccheri quadrilateral $ABCD$ with base $\overline{AB}$ and summit $\overline{CD}$. The summit and the base are parallel by Corollary 3.6.5. The two legs are parallel by the Alternate Interior Angle Theorem. Thus $ABCD$ is a parallelogram and hence by assumption $\angle A \cong \angle C$ and $\angle B \cong \angle D$. This means that $ABCD$ is a rectangle. However, we have shown in part (e) that a rectangle need not exist in neutral geometry, yielding a contradiction.

(o) The diagonals of a rectangle are congruent.

True. By Theorem 3.6.8 any two opposite sides in a rectangle must be congruent. Thus a rectangle is a Saccheri quadrilateral and hence its diagonals are congruent by Theorem 3.6.1.

(p) The length of the side between two right angles in a lambert quadrilateral is less than or equal to the length of the opposite side.

True by Theorem 3.6.8.

(q) Similar but not congruent triangles exist.

False. Indeed, suppose there exist two similar non-congruent triangles then there exist (think about why) a triangle $\triangle ABC$ and some points B', C' with $A - B' - B$,

$A - C' - C$ such that the triangles $\triangle ABC$ and $\triangle AB'C'$ are similar and $\angle 1 \cong \angle 2$, and $\angle 3 \cong \angle 4$.

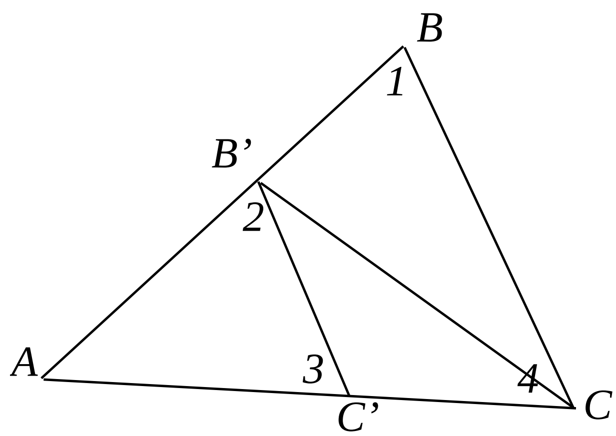

FIGURE 1. Figure for part (q)

Then the angle sum in the quadrilateral $BB'C'C$ is

$$
\begin{aligned}
&m\angle 1 + (180^\circ - m\angle 2) + (180^\circ - m\angle 3) + m\angle 4 = \\
&m\angle 1 + (180^\circ - m\angle 1) + (180^\circ - m\angle 4) + m\angle 4 = 360^\circ.
\end{aligned}
$$

Draw the diagonal $\overline{B'C}$. Each of the triangles $\triangle B'C'C$, $\triangle B'BC$ has angle sum at most 180° by the Saccheri-Legendre Theorem. However their angle sums add up to the angle sum of the quadrilateral $BB'C'C$, that is 360°. Therefore each of the triangles $\triangle B'C'C$, $\triangle B'BC$ has angle sum exactly 180°. By Corollary 3.6.16 this implies that all triangles have angle sum 180° which in turn by Theorem 3.6.18 implies that the Euclidean parallel postulate holds. This is not true in neutral geometry, yielding a contradiction.