Problem 1.
Let $y_1, y_2, y_3 : \mathbb{R} \to \mathbb{R}$ be functions which are linearly dependent on $(-\infty, \infty)$.
For each of the following statements indicate if it is true or false:

(1) There exist $i \neq j$, where $i, j \in \{1, 2, 3\}$, and a number $c \in \mathbb{R}$ such that $y_j(x) = cy_i(x)$ for all $x \in \mathbb{R}$.

(2) For every $x \in \mathbb{R}$ we have $W(y_1, y_2, y_3)(0) = 0$.

(3) We have $W(y_1, y_2, y_3)(0) = 0$.

(4) If $c_1, c_2, c_3 \in \mathbb{R}$ are such that $c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) = 0$, then $c_1 = c_2 = c_3 = 0$.

(5) If $c_1, c_2, c_3 \in \mathbb{R}$ are such that for all $x \in \mathbb{R}$ $c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) = 0$, then for some $i \in \{1, 2, 3\}$ $c_i \neq 0$.

(6) There exist $c_1, c_2, c_3 \in \mathbb{R}$ are such that $c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) = 0$ and that for all $x \in \mathbb{R}$ $c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) = 0$.

(7) The functions $y_1, y_2$ are linearly dependent on $(-\infty, \infty)$.

(8) The functions $y_1, y_2$ are linearly independent on $(-\infty, \infty)$.

(9) For any functions $y_4 : \mathbb{R} \to \mathbb{R}$ the functions $y_1, y_2, y_3, y_4$ are linearly dependent on $(-\infty, \infty)$.

Answers:

(1) False. For example, $y_1 = x$, $y_2 = x^2$ and $y_3 = x + x^2$ are linearly dependent on $\mathbb{R}$ since $y_1 + y_2 - y_3 \equiv 0$. However, none of the functions $x, x^2, x + x^2$ is a scalar multiple of one of the others.

(2) False. For example, for $y_1 = y_2 = y_3 = |x|$ these functions are linearly dependent on $\mathbb{R}$. However, $W(y_1, y_2, y_3)(0)$ does not exist since $|x|$ is not differentiable at $x = 0$.

(3) False, for the same reason as in (2).

(4) False, since the functions $y_1, y_2, y_3$ are assumed to be linearly dependent on $(-\infty, \infty)$.

(5) False. For example, if we take $c_1 = c_2 = c_3 = 0$, then $c_1 y_1 + c_2 y_2 + c_3 y_3 \equiv 0$, regardless of which functions $y_1, y_2, y_3$ we use.

(6) True, by definition of linear dependence.

(7) False. For example, $y_1 = x$, $y_2 = x^2$ and $y_3 = x + x^2$ are linearly dependent on $\mathbb{R}$, but the functions $y_1 = x$, $y_2 = x^2$ are linearly independent on $\mathbb{R}$.

(8) False. For example, if we take $y_1 = y_2 = y_3 = x$, then $y_1, y_2, y_3$ are linearly dependent on $\mathbb{R}$ and $y_1, y_2$ are linearly dependent on $\mathbb{R}$.

(9) True. Since $y_1, y_2, y_3$ are assumed to be linearly dependent on $\mathbb{R}$, there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that at least one of $c_1, c_2, c_3$ is $\neq 0$ but $c_1 y_1 + c_2 y_2 + c_3 y_3 \equiv 0$ on $\mathbb{R}$. For any function $y_4$ we can take $c_4 = 0$ and then
\[ c_1y_1 + c_2y_2 + c_3y_3 \equiv 0 \text{ on } \mathbb{R}, \text{ so that } y_1, y_2, y_3, y_4 \text{ are linearly dependent on } \mathbb{R}. \]

**Problem 2.**

Let \( f : \mathbb{R} \to \mathbb{R} \) be the 20-periodic function such that
\[
f(t) = \begin{cases} 
-1, & \text{if } -10 < t \leq 3 \\
\frac{t50}{10}, & \text{if } 3 < t \leq 10 
\end{cases}
\]
and let
\[
f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{\pi nt}{10} + b_n \sin \frac{\pi nt}{10} \right)
\]
be the General Fourier Series of \( f \).

For each of the following statements indicate if it is true or false:

1. For every \( t \in (-20, 55) \) the series
\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{\pi nt}{10} + b_n \sin \frac{\pi nt}{10} \right)
\]
converges.

2. For every integer \( n \geq 1 \) we have
\[
b_n = \frac{1}{10} \int_{-97}^{97} f(t) \sin \frac{\pi nt}{10} \, dt
\]

3. For every integer \( n \geq 1 \) we have
\[
b_n = \frac{2}{10} \int_{0}^{10} f(t) \sin \frac{\pi nt}{10} \, dt
\]

4. We have
\[
-1 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{3\pi n}{10} + b_n \sin \frac{3\pi n}{10} \right).
\]

5. We have
\[
\frac{350}{2} - 1 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{3\pi n}{10} + b_n \sin \frac{3\pi n}{10} \right).
\]

6. We have
\[
-1 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{42\pi n}{10} + b_n \sin \frac{42\pi n}{10} \right).
\]

7. For every \( t \in (-2, 3) \) we have
\[
0 = \sum_{n=1}^{\infty} \frac{\pi n}{10} \left( -a_n \sin \frac{\pi nt}{10} + b_n \cos \frac{\pi nt}{10} \right).
Answers:

1. True. Since the function $f$ is piecewise-smooth, by the Convergence
   Theorem the General Fourier Series of $f$ converges for every $t \in \mathbb{R}$, and, in
   particular, for every $t \in (-20, 55)$.

2. True. If $g(t)$ is a piecewise-continuous $p$-periodic function (where
   $p > 0$) then for any $a, b \in \mathbb{R}$ we have $\int_{a}^{a+p} g(t) \, dt = \int_{b}^{b+p} g(t) \, dt$. In our case
   the function $f(t) \sin \frac{\pi nt}{10}$ is 20-periodic and hence
   \[
   b_n = \frac{1}{10} \int_{-10}^{10} f(t) \sin \frac{\pi nt}{10} \, dt = \frac{1}{10} \int_{-77}^{97} f(t) \sin \frac{\pi nt}{10} \, dt
   \]
   (3) False, since the function $f(t)$ in this example is not odd.

3. False. The function $f(t)$ is discontinuous at $t = 3$, with $f(3+) = 3^{50}$
   and $f(3-) = -1$. Therefore, by the Convergence Theorem,
   \[
   \frac{3^{50} - 1}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{3\pi n}{10} + b_n \sin \frac{3\pi n}{10} \right).
   \]

4. True; see explanation for (4) above.

5. True. We have $42 = 2 + 2 \cdot 20$. Therefore, by the Convergence Theorem,
   \[
   \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{42\pi n}{10} + b_n \sin \frac{42\pi n}{10} \right) = f(42) = f(2) = -1.
   \]

Problem 3.

Consider the initial value problem
\[
(*) \quad \frac{dy}{dx} = \frac{1}{4-x} \sin(y^2 e^{xy} + 2xy - 5), \quad y(1) = 55.
\]

For each of the following statements indicate if it is true or false:

1. The problem is guaranteed to have a unique solution on the interval
   $(0, 4)$.
2. The problem is guaranteed to have a solution on the interval $(0, 4)$.
3. There exists $\epsilon > 0$ such that the problem has a solution on the
   interval $(1 - \epsilon, 1 + \epsilon)$.
4. There exists $\epsilon > 0$ such that the problem has at most one solution
   on the interval $(1 - \epsilon, 1 + \epsilon)$.
5. For some $\epsilon > 0$ the problem has a solution
   on the interval $(1 - 2\epsilon, 1 + 3\epsilon)$.

Answers

1. False. Theorem 1 on p. 24 never guarantees the existence of a solution
   on a specific interval, such as the interval $(0, 4)$. 

(2) False, for the same reason as in (1).
(3) True, by Theorem 1 on p. 24.
(4) True, by Theorem 1 on p. 24.
(5) True, by Theorem 1 on p. 24.

**Problem 4.**

Consider the problem

\[
\begin{cases}
    y_{tt} = 25y_{xx} & \text{for } t > 0, \ 0 < x < \pi, \\
    y(0, t) = y(\pi, t) = 0, & \text{for } t > 0, \\
    y(x, 0) = x^2 - \pi x & \text{for } 0 < x < \pi, \\
    y_t(x, 0) = 0, & \text{for } 0 < x < \pi.
\end{cases}
\]

For each of the following statements indicate if it is true or false:

(1) The function

\[
y(x, t) = \frac{1}{2} \left[ (x + 5t)^2 + (x + 5t) + (x - 5t)^2 + (x - 5t) \right],
\]

where \( t \geq 0, \ 0 \leq x \leq \pi \), is a solution of (**).

(2) If \( y(x, t) \) is the solution of (**), then

\[
y(1, 1) = \frac{23}{2}.
\]

(3) Problem (**) has a solution of the form

\[
y(x, t) = \sum_{n=1}^{\infty} c_n \cos(5nt) \sin(nx)
\]

for some choice of constants \( c_n \in \mathbb{R}, \ n = 1, 2, 3, \ldots \).

(4) Problem (**) has the solution

\[
y(x, t) = \sum_{n=1}^{\infty} c_n \cos(5nt) \sin(nx)
\]

where

\[
c_n = \int_0^{\pi} (x^2 - \pi x) \sin(nx) \, dx
\]

for \( n = 1, 2, 3, \ldots \).

(5) Problem (**) has the solution

\[
y(x, t) = \sum_{n=1}^{\infty} c_n \cos(5nt) \sin(nx)
\]

where

\[
c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 - \pi x) \sin(nx) \, dx
\]

for \( n = 1, 2, 3, \ldots \).
(6) Problem (**) has the solution
\[ y(x, t) = \sum_{n=1}^{\infty} c_n \cos(5nt) \sin(nx) \]
where \[ c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_O(x) \sin(nx) \, dx \]
for \( n = 1, 2, 3, \ldots \), and where \( f_O(x) \) is the \( 2\pi \)-periodic odd extension of the function \( f(x) = x^2 - \pi x, \, 0 < x < \pi \).

(7) Problem (**) has the solution
\[ y(x, t) = \sum_{n=1}^{\infty} c_n \cos(5nt) \sin(nx) \]
where \[ c_n = \frac{1}{\pi} \int_{10}^{10+2\pi} f_O(x) \sin(nx) \, dx \]
for \( n = 1, 2, 3, \ldots \), and where \( f_O(x) \) is the \( 2\pi \)-periodic odd extension of the function \( f(x) = x^2 - \pi x, \, 0 < x < \pi \).

Answers:
(1) False. Note that for \( f(x) = x^2 - \pi x, \, 0 \leq x \leq \pi \), we have \( f(0) = f(\pi) = 0 \). Therefore the odd \( 2\pi \)-periodic extension \( f_O(x) \) of \( f \) is continuous on \( \mathbb{R} \). The D’Alambert’s form of the solution of (**) is \( y = \frac{1}{2}[f_O(x+5t) + f_O(x-5t)] \). However, \( f_O(x) \) is defined by different formulas from \( x^2 - \pi x \) outside of the interval \( [0, \pi] \) and it is not true that for arbitrary \( x \in \mathbb{R} \) and \( t \geq 0 \) \( \frac{1}{2}[f_O(x+5t) + f_O(x-5t)] = \frac{1}{2}[(x+5t)^2 - \pi(x+5t) + (x-5t)^2 - (x-5t)] \).

(2) False.
As explained in part (1) above, by D’Alambert’s formula, we have \( y = \frac{1}{2}[f_O(x+5t) + f_O(x-5t)] \). In particular, \( y(1, 1) = \frac{1}{2}[f_O(6) + f_O(-4)] \). Since \( f_O \) is \( 2\pi \)-periodic and \( -\pi < 6 - 2\pi < 0 \), we have \( f_O(6) = f_O(6-2\pi) = -f(2\pi - 6) = -(2\pi - 6)^2 + \pi(2\pi - 6) = -2\pi^2 + 18\pi - 36 \). Also, \( 0 < -4 + 2\pi < \pi \) and hence \( f_O(-4) = f_O(-4+2\pi) = f(-4+2\pi) = (2\pi - 4)^2 - \pi(2\pi - 4) = 2\pi^2 - 12\pi + 16 \). Therefore
\[ y(1, 1) = \frac{1}{2}[f_O(6) + f_O(-4)] = -2\pi^2 + 3\pi - 10 \neq \frac{23}{2} \].

(3) True.

(4) False. The factor of \( \frac{2}{\pi} \) in front of the integral sign in the formula for \( c_n \) is missing.

(5) False. See item (6) below. Note that \( f_O(x) = -x^2 + \pi x \) for \( -\pi < x < 0 \).
(6) True. Since \( f_O \) is odd, we have
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f_O(x) \sin(nx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} f_O(x) \sin(nx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} (x^2-\pi x) \sin(nx) \, dx.
\]

(7) True. Since \( f_O \) is 2\( \pi \)-periodic, we have
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f_O(x) \sin(nx) \, dx = \frac{1}{\pi} \int_{10}^{10+2\pi} f_O(x) \sin(nx) \, dx.
\]

**Problem 5.**

Consider the equation
\[
y''' + 3y'' + 3y' + y = x^2 e^{-x} + 5xe^{-x} \cos(2x)
\]
on the interval \((-\infty, \infty)\).

For each of the following statements indicate if it is true or false:

(1) Equation (†) has a particular solution of the form
\[
y = (A + Bx + Cx^2)e^{-x} + (D + Ex)e^{-x} \cos(2x)
\]
for some constants \(A, B, C, D, E\).

(2) Equation (†) has a particular solution of the form
\[
y = x^2(A + Bx + Cx^2)e^{-x} + (D + Ex)e^{-x} \cos(2x)
\]
for some constants \(A, B, C, D, E\).

(3) Equation (†) has a particular solution of the form
\[
y = x^3(A + Bx + Cx^2)e^{-x} + (D + Ex)e^{-x} \cos(2x)
\]
for some constants \(A, B, C, D, E\).

(4) Equation (†) has a particular solution of the form
\[
y = x^3[(A + Bx + Cx^2)e^{-x} + (D + Ex)e^{-x} \cos(2x)]
\]
for some constants \(A, B, C, D, E\).

(5) Equation (†) has a particular solution of the form
\[
y = x^3(A + Bx + Cx^2)e^{-x} + (D + Ex)e^{-x} \cos(2x) + (F + Gx)e^{-x} \sin(2x)
\]
for some constants \(A, B, C, D, E, F, G\).

(6) For every \(A, B, C, D, E, F, G \in \mathbb{R}\) the function
\[
y = x^3(A + Bx + Cx^2)e^{-x} + (D + Ex)e^{-x} \cos(2x) + (F + Gx)e^{-x} \sin(2x)
\]
is a solution of equation (†).

(7) The general solution of of equation (†) is
\[
y = (c_1+c_2x+c_3x^2)e^{-x} + x^3(A+Bx+Cx^2)e^{-x} + (D+Ex)e^{-x} \cos(2x) + (F+Gx)e^{-x} \sin(2x)
\]
where \(c_1, c_2, c_3, A, B, C, D, E, F, G \in \mathbb{R}\) are arbitrary constants.
(8) There exist $A_0, B_0, C_0, D_0, E_0, F_0, G_0 \in \mathbb{R}$ such that the general solution of of equation (†) is
\[ y = (c_1 + c_2 x + c_3 x^2) e^{-x} + x^2 (A_0 + B_0 x + C_0 x^2) e^{-x} + (D_0 + E_0 x) e^{-x} \cos(2x) + (F_0 + G_0 x) e^{-x} \sin(2x) \]
where $c_1, c_2, c_3 \in \mathbb{R}$ are arbitrary constants.

**Answers:**

(1) False. Duplication with terms from $y_c$ is not yet eliminated, and the terms involving $e^{-x} \sin(2x)$ are missing.

(2) False, for similar reasons to (1).

(3) False. Duplication with terms from $y_c$ is eliminated, but the terms involving $e^{-x} \sin(2x)$ are still missing.

(4) False. See (5) below for the correct form of $y_p$.

(5) True.

(6) False. To make the statement in (5) true we must replace the opening phrase “For every” by “For some”.

(7) False, for similar reasons to (6). The constants $A, B, C, D, E, F, G$ are not allowed to be arbitrary.

(8) True.