(b) Because for $x$ fixed the temperature is a function of the product $kt$, in the case of concrete slabs with $k = 0.005$ the same temperature will be attained when

$$(0.005)(t) = (0.15)(1800),$$

that is, when $t = 54000 \text{ sec} = 15 \text{ hr}.$

SECTION 9.6

VIBRATING STRINGS AND THE ONE-DIMENSIONAL WAVE EQUATION

In Problems 1–10 we use the general solution

$$y(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi at}{L} + B_n \sin \frac{n\pi at}{L} \right) \sin \frac{n\pi x}{L} \quad (*)$$

of the string equation $y_{tt} = a^2 y_{xx}$ with endpoint conditions $y(0, t) = y(L, t) = 0$. This form of the solution is obtained by superposition of the solutions in Equations (23) and (33) of Problems A and B in this section. It remains only to choose the coefficients $\{A_n\}$ and $\{B_n\}$ so as to satisfy given initial conditions

$$y(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = f(x), \text{ thus, } A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx; \text{ and}$$

$$y_t(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} B_n \sin \frac{n\pi x}{L} = g(x), \text{ thus, } B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx.$$

1. Here $a = 2$ and $L = \pi$. To satisfy the condition $y(x, 0) = (1/10) \sin 2x$ we choose $A_2 = 1/10$ in Eq. (*) above, and $A_n = 0$ otherwise. To satisfy the condition $y_t(x, 0) = 0$ we choose $B_n = 0$ for all $n$. Thus

$$y(x, t) = \frac{1}{10} \cos 4t \sin 2x.$$

2. Here $a = L = 1$. To satisfy the condition

$$y(x, 0) = \frac{1}{10} \sin \pi x - \frac{1}{20} \sin 3\pi x$$

we choose $A_1 = 1/10$ and $A_3 = -1/20$ in Eq. (*) above, and $A_n = 0$ otherwise. To satisfy the condition $y_t(x, 0) = 0$ we choose $B_n = 0$ for all $n$. Thus
\[ y(x, t) = \frac{1}{10} \cos \pi x \sin \pi t - \frac{1}{20} \cos 3\pi x \sin 3\pi t. \]

3. Here \( a = 1/2 \) and \( L = \pi \). Choosing \( A_1 = 1/10 \) and \( A_n = 0 \) otherwise, \( B_1 = 1/5 \) and \( B_n = 0 \) otherwise, we get
\[ y(x, t) = \frac{1}{10} \left( \cos \frac{t}{2} + 2 \sin \frac{t}{2} \right) \sin x. \]

4. Here \( a = 1/2 \) and \( L = 2 \), so \( n\pi L = n\pi x/2 \) and \( n\pi x/L = n\pi x/4 \). To satisfy the condition
\[ y(x, 0) = \frac{1}{5} \sin \pi x \cos \pi x = \frac{1}{10} \sin 2\pi x = \frac{1}{10} \sin \frac{4\pi x}{2}, \]
we choose \( A_4 = 1/10 \) and \( A_n = 0 \) for \( n \neq 4 \). To satisfy the condition \( y(x, 0) = 0 \) we choose \( B_n = 0 \) for all \( n \). Thus
\[ y(x, t) = \frac{1}{10} \cos \pi x \sin 2\pi x. \]

5. Here \( a = 5 \) and \( L = 3 \). Choosing \( A_3 = 1/4 \) and \( A_n = 0 \) for \( n \neq 3 \), \( B_6 = 1/\pi \) and \( B_n = 0 \) for \( n \neq 6 \), we get
\[ y(x, t) = \frac{1}{4} \cos 5\pi x \sin \pi x + \frac{1}{\pi} \sin 10\pi x \sin 2\pi x. \]

6. Here \( a = 10 \) and \( L = \pi \). To satisfy the condition \( y(x, 0) = 0 \) we choose \( B_n = 0 \) for all \( n \), so
\[ y(x, t) = \sum_{n=1}^{\infty} A_n \cos 10nt \sin nx. \]

To satisfy the condition \( y(x, 0) = x(\pi - x) \) we choose
\[ A_n = \frac{2}{\pi} \int_{0}^{\pi} x(\pi - x) \sin nx \, dx = \frac{4 - 4 \cos n\pi - 2n\pi \sin n\pi}{n^3 \pi} = \begin{cases} \frac{8}{n^3 \pi} & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ odd}. \end{cases} \]

This gives the solution
\[ y(x, t) = \frac{8}{\pi} \sum_{n \text{ odd}} \frac{\cos 10nt \sin nx}{n^3}. \]

7. Here \( a = 10 \) and \( L = 1 \). To satisfy the condition \( y(x, 0) = 0 \) we choose \( A_n = 0 \) for all \( n \), so

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\[ y(x, t) = \sum_{n=1}^{\infty} B_n \sin 10n\pi t \sin n\pi x. \]

To satisfy the condition \( y(x, 0) = x \) we choose

\[ B_n = \frac{1}{10n\pi} \cdot \frac{2(-1)^{n+1}}{n\pi} = \frac{(-1)^{n+1}}{5n^2 \pi^2} \]

for \( n \geq 1 \) (see Equation (16) in Section 9.3). This gives

\[ y(x, t) = \frac{1}{5\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \sin 10n\pi t \sin n\pi x. \]

8. Here \( a = 2 \) and \( L = \pi \). To satisfy the condition \( y(x, 0) = \sin x \) we choose \( A_1 = 1 \) and \( A_n = 0 \) for \( n > 1 \), so

\[ y(x, t) = \cos 2t \sin x + \sum_{n=1}^{\infty} B_n \sin 2nt \sin nx, \text{ so} \]

\[ y(x, t) = -2 \sin 2t \sin x + \sum_{n=1}^{\infty} 2nB_n \cos 2nt \sin nx. \]

The condition \( y(x, 0) = 1 \) will be satisfied if \( 2nB_n = 4/\pi n \) for \( n \) odd and \( b_n = 0 \) for \( n \) even. We therefore choose \( B_n = 2/\pi n^2 \) for \( n \) odd and \( B_n = 0 \) for \( n \) even, so

\[ y(x, t) = \cos 2t \sin x + \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin 2nt \sin nx}{n}. \]

9. Here \( a = 2 \) and \( L = 1 \). To satisfy the condition \( y(x, 0) = 0 \) we choose \( A_n = 0 \) for all \( n \), so

\[ y(x, t) = \sum_{n=1}^{\infty} B_n \sin 2n\pi t \sin n\pi x. \]

To satisfy the condition \( y(x, 0) = x(1-x) \) we choose

\[ B_n = \frac{1}{n\pi} \int x(1-x) \sin n\pi x \, dx = \frac{2-2\cos n\pi - n\pi \sin n\pi}{n^4 \pi^4}. \]

Hence

\[ y(x, t) = \frac{4}{\pi^4} \sum_{n \text{ odd}}^{\infty} \frac{\sin 2n\pi t \sin n\pi x}{n^4}. \]

10. Here \( a = 5 \) and \( L = \pi \) so

\[ y(x, t) = \sum_{n=1}^{\infty} (A_n \cos 5nt + B_n \sin 5nt) \sin nx. \]
We first compute the Fourier sine series \( \sin^2 x = \sum_{n=1}^{\infty} b_n \sin nx \) and find that \( b_n = 0 \) if \( n \) is even whereas

\[
b_n = \frac{2}{\pi} \int_0^\pi \sin^2 x \sin nx \, dx = \frac{4(\cos n\pi - 1)}{\pi n(n^2 - 4)} = \frac{8}{\pi n(4-n^2)}
\]

if \( n \) is odd. To satisfy the condition \( y(x, t) = \sin^2 x \) we choose \( A_n = b_n \), and to satisfy the condition \( y(x, t) = \sin^2 x \) we choose \( B_n = b_n/5n \). Then

\[
y(x, t) = \frac{8}{5\pi} \sum_{n \text{ odd}} \frac{(5n \cos 5nt + \sin 5nt) \sin nx}{n^2(4-n^2)}.
\]

11. Substitution of \( L = 2 \text{ ft, } T = 32 \text{ lb, } \) and the linear density

\[
\rho = \frac{1/32 \text{ oz}}{2 \text{ ft}} = \frac{1 \text{ oz}}{64 \text{ ft}} \cdot \frac{1 \text{ lb}}{16 \text{ oz}} \cdot \frac{1 \text{ slug}}{32 \text{ lb}} = \frac{1 \text{ slug}}{32^2 \text{ ft}}
\]

in Eqs. (2) and (26) in the text yields the velocity \( a = \sqrt{T/\rho} = \sqrt{32^4} = 1024 \text{ ft/sec} \) with which waves move along the string, and its fundamental frequency

\[
\nu_1 = \frac{1}{2L} \sqrt{\frac{T}{\rho}} = \frac{a}{2L} = 256 \text{ Hz},
\]

which is approximately middle C.

12. The value of

\[
y(x, t) = \frac{4v_0L}{\pi^2a} \sum_{n \text{ odd}} \frac{1}{n^2} \sin \frac{n\pi at}{L} \sin \frac{n\pi x}{L}
\]

is maximal when each of the sine products is 1. This happens when \( x = L/2, \ t = L/2a \):

\[
y\left(\frac{L}{2}, \frac{L}{2a}\right) = \frac{4v_0L}{\pi^2a} \sum_{n \text{ odd}} \frac{1}{n^2} \sin^2 \frac{n\pi}{2} = \frac{4v_0L}{\pi^2a} \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{4v_0L}{\pi^2a} \cdot \frac{\pi^2}{8} = \frac{v_0L}{2a}.
\]

Using fps units with the string of Problem 11 where \( L = 2 \text{ ft, } a = 1024 \text{ ft/sec, } \) and \( v_0 = 60 \text{ mph} = 88 \text{ ft/sec, } \) we get

\[
y'_{\text{max}} = \frac{88 \times 2}{2 \times 1024} \approx 0.0859 \text{ ft} \approx 1 \text{ inch}.
\]

13. If \( y(x, t) = F(x + at) = F(u) \) with \( u = x + at, \) then the chain rule gives

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\[
\frac{\partial y}{\partial x} = \frac{dF}{du} \frac{\partial u}{\partial x} = F'(u) \cdot 1 = F'(x+at);
\]
\[
\frac{\partial y}{\partial t} = \frac{dF}{du} \frac{\partial u}{\partial t} = F'(u) \cdot a = a F'(x+at) = a \frac{\partial y}{\partial x};
\]
\[
\frac{\partial^2 y}{\partial x^2} = \frac{dF'}{du} \frac{\partial u}{\partial x} = F''(u) \cdot 1 = F''(x+at);
\]
\[
\frac{\partial^2 y}{\partial t^2} = a \frac{dF''}{du} \frac{\partial u}{\partial t} = a F''(u) \cdot a = a^2 F''(x+at) = a^2 \frac{\partial^2 y}{\partial x^2}.
\]

14. \quad y(0,t) = \frac{1}{2} [F(at) + F(-at)] = \frac{1}{2} [F(at) - F(at)] = 0
\[
y(L,t) = \frac{1}{2} [F(L+at) + F(L-at)]
\]
\[
= \frac{1}{2} [F(L+at) - F(-L+at)] = \frac{1}{2} [F(2L+(-L+at)) - F(-L+at)] = 0
\]
\[
y(x,0) = \frac{1}{2} [F(x) + F(x)] = F(x)
\]
\[
y_i(x,t) = \frac{1}{2} [a F'(x+at) - a F'(x-at)]
\]
\[
y_i(x,0) = \frac{1}{2} [a F'(x) - a F'(x)] = 0
\]

15. If \( y(x,0) = 0 \) then the fundamental theorem of calculus gives
\[
y(x,t) = y(x,t) - y(x,0) = \int_0^t y_i(x,\tau) \, d\tau = \int_0^t \frac{1}{2} [G(x+a\tau) + G(x-a\tau)] \, d\tau.
\]

16. If \( u = x+at, \ v = x-at \) then we solve readily for \( x = \frac{1}{2} (u+v), t = \frac{1}{2a} (u-v) \). Hence
\[
\frac{\partial y}{\partial u} = \frac{\partial}{\partial u} \left[ y \left( \frac{1}{2} (u+v), \frac{1}{2a} (u-v) \right) \right] = \frac{\partial y}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial y}{\partial t} \frac{\partial t}{\partial u} = \frac{1}{2a} \frac{\partial y}{\partial x} + \frac{1}{2a} \frac{\partial y}{\partial t};
\]
\[
\frac{\partial y}{\partial v} = \frac{\partial}{\partial v} \left[ y \left( \frac{1}{2} (u+v), \frac{1}{2a} (u-v) \right) \right] = \frac{\partial y}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial t} \frac{\partial t}{\partial v} = \frac{1}{2a} \frac{\partial y}{\partial x} - \frac{1}{2a} \frac{\partial y}{\partial t};
\]
\[
\frac{\partial^2 y}{\partial v \partial u} = \frac{\partial}{\partial v} \left( \frac{1}{2a} \frac{\partial y}{\partial x} + \frac{1}{2a} \frac{\partial y}{\partial t} \right)
\]
\[
= \frac{1}{2a} \frac{\partial}{\partial x} \left( \frac{1}{2a} \frac{\partial y}{\partial x} + \frac{1}{2a} \frac{\partial y}{\partial t} \right) - \frac{1}{2a} \frac{\partial}{\partial t} \left( \frac{1}{2a} \frac{\partial y}{\partial x} + \frac{1}{2a} \frac{\partial y}{\partial t} \right)
\]
\[
= \frac{1}{4} \frac{\partial^2 y}{\partial x^2} + \frac{1}{4a} \frac{\partial^2 y}{\partial x \partial t} - \frac{1}{4} \frac{\partial^2 y}{\partial x^2} - \frac{1}{4} \frac{\partial^2 y}{\partial t^2} = \frac{1}{4a^2} \left( \frac{\partial^2 y}{\partial x^2} - a^2 \frac{\partial^2 y}{\partial t^2} \right) = 0.
\]

Now if \( \frac{\partial^2 y}{\partial v \partial u} = \frac{\partial}{\partial v} \left( \frac{\partial y}{\partial u} \right) = 0 \) then antidifferentiation with respect to \( v \) gives