1.

(a) Consider the vector space $V = \mathbb{P}_2[t]$ of polynomials in $t$ of degree $\leq 2$ with the basis $B = 1, t + 1$. [You do not need to prove that $B$ is a basis of $V$.] For $f(t) = 2t + 3$, compute the coordinate vector $[f]_B \in \mathbb{R}^2$.

Answer:
We have $t = (t + 1) - 1$ and therefore $f = 2t + 3 = 2(t + 1) - 2 + 3 = 2(t + 1) + 1 = 1 \cdot 1 + 2 \cdot (t + 1)$.

Hence $[f]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Consider the matrix $A = \begin{bmatrix} 1 & 6 & 4 & 0 & 7 & 5 \\ 0 & 0 & 1 & -9 & 0 & 1 \\ 0 & 0 & 0 & 0 & -8 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Find $\dim \text{Col}(A)$ and $\dim \text{Null}(A)$.

Answer:
The matrix $A$ is already in row-echelon form. The number of pivot columns is 3 (namely, columns number 1, 3, 4). Therefore $\dim \text{Col}(A) = 3$. The number of pivot-free columns is also 3. Therefore $\dim \text{Null}(A) = 3$ as well.

(c) If $B = \vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis of a subspace $H$ of a vector space $V$, and if $\vec{u} \in V$ is such that $\vec{u} \notin H$, does it follow that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{u}$ are linearly independent?

If yes, explain why, and if not, give a counter-example.

Answer:
Yes, $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{u}$ are linearly independent. Indeed, suppose that $c_1, c_2, c_3, c_4 \in \mathbb{R}$ are such that

\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{u} = \vec{0}. \]

We claim that $c_1 = c_2 = c_3 = c_4 = 0$.

If $c_4 = 0$ then $c_1 = c_2 = c_3 = 0$ since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are a basis of $H$ are thus are linearly independent. In this case $c_1 = c_2 = c_3 = c_4 = 0$, as required.

Suppose that $c_4 \neq 0$. Then $\vec{u} = -\frac{c_4}{c_4} \vec{v}_1 - \frac{c_2}{c_4} \vec{v}_2 - \frac{c_3}{c_4} \vec{v}_3 \in H$, contradicting the assumption that $\vec{u} \notin H$. Thus the case $c_4 \neq 0$ is not possible, and therefore $c_1 = c_2 = c_3 = c_4 = 0$, by the previous case.

Hence the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{u}$ are linearly independent.