Diagnostic Quiz 15. October 24, 2017

1. 

(a) Let $V = \mathbb{R}^2$ and define $(x, y) \oplus (x', y') := (0, 0)$ and $c \circ (x, y) := (0, 0)$ for arbitrary $x, y, x', y', c \in \mathbb{R}$. Is $V$ with the operations $\oplus, \circ$ a vector space? Why or why not?

Answer:
No, $V$ is not a vector space. For example, the axiom $1 \circ \vec{v} = \vec{v}$ for every $\vec{v} \in V$ is not satisfied since, for example, $1 \circ (2, 7) = (0, 0) \neq (2, 7)$.

(b) Consider the vector space $F(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$, with pointwise addition and multiplication by a scalar [You do not have to prove that $F(\mathbb{R}, \mathbb{R})$ is a vector space.]

Let $H = \{f(x) = ax + b | a, b \in \mathbb{R}\} \subseteq F(\mathbb{R}, \mathbb{R})$. Is $H$ a subspace of $F(\mathbb{R}, \mathbb{R})$? Why or why not?

Answer:
Yes, $H$ is a subspace of $F(\mathbb{R}, \mathbb{R})$. Indeed, if $f = ax + b \in H$ and $g = a'x + b' \in H$ then $f + g = (a + a')x + (b + b') \in H$. Also, if $f = ax + b \in H$ and $c \in \mathbb{R}$ then $cf = cac + cb \in H$.

(c) Let $F(\mathbb{R}, \mathbb{R})$ be as in part (b). Is it true that $\sin x \in \text{Span}(x, x^2)$ in $F(\mathbb{R}, \mathbb{R})$? Why or why not?

Answer:
No, $\sin x \not\in \text{Span}(x, x^2)$ in $F(\mathbb{R}, \mathbb{R})$. Indeed, suppose that $\sin x \in \text{Span}(x, x^2)$. Then there exist $s_1, s_2 \in \mathbb{R}$ such that $\sin x = s_1 x + s_2 x^2$ for all $x \in \mathbb{R}$. Differentiating this equation once, we get $\cos x = s_1 + 2s_2 x$, and then differentiating again we get $-\sin x = 2s_2$ and then differentiating the third time we get $-\cos x = 0$ for all $x \in \mathbb{R}$, yielding a contradiction.