1.

(a) Let $A$ be a $3 \times 3$ invertible matrix with the columns $A = [a_1 | a_2 | a_3]$. Consider the linear system

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a_1.$$ 

What can we say about $x_2$ and $x_3$?

**Answer:**

By Cramer’s rule, we have

$$x_2 = \frac{1}{\det(A)} \det[a_1 | a_1 | a_3] = 0$$

and similarly

$$x_3 = \frac{1}{\det(A)} \det[a_1 | a_2 | a_1] = 0.$$

(b) Let $A = \begin{bmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & c & * \\ 0 & 0 & 0 & d \end{bmatrix}$ where $a, b, c, d \in \mathbb{R}$ are nonzero real numbers.

(So that $\det(A) = abcd \neq 0$ and $A$ is invertible).

Is it true that $A^{-1}$ is again an upper-triangular matrix? Why or why not?

**Answer:**

Yes, $A^{-1}$ is an upper-triangular matrix. We can make this conclusion using the formula $A^{-1} = \frac{1}{\det(A)} \adj(A)$ for the inverse of $A$. In the adjugate matrix $\adj(A)$ every entry in the position $ij$ with $i > j$ (below the diagonal) is computed as $C_{ij} = (-1)^{i+j} A_{ji}$ for the original matrix $A$. Computing the determinants $A_{ji}$ for $j < i$ for $A$ shows that they all produce $2 \times 2$ matrices with at least one row or column of 0s, so that $C_{ji} = 0$.

(c) Compute the area of the parallelogram in $\mathbb{R}^2$ with the sides given by the segments $AB$ and $AC$ where $A = (0, 2)$, $B = (-1, 3)$, $C = (4, 1)$.

**Answer:**

Let $\vec{v}_1 = A \vec{c} = (-1, 1)$ and $\vec{v}_2 = A \vec{c} = (4, -1)$. Form the matrix

$$M = [\vec{v}_1 | \vec{v}_2] = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}.$$ 

Then the area of the parallelogram in question is $|\det(M)| = |-3| = 3$. 
