Math 225  Midterm Exam 1 (Solutions)
Sections SL1 and TL1 (Prof. Ilya Kapovich)
Thursday, September 28, 2017

Problem 1. [10 points]
(a) For each of the following matrices, indicate whether the matrix is in
• echelon form but not reduced echelon form
• reduced echelon form
• not in echelon form
[No explanation is required.]
\[
\begin{pmatrix}
7 & 0 & 0 & 8 & 0 \\
0 & 0 & 1 & \pi & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\text{ not in echelon form}
\]
\[
\begin{pmatrix}
5 & -5 & 0 & 7 \\
0 & 0 & -11 & 3 \\
0 & 0 & 0 & 1
\end{pmatrix}
\text{ echelon form but not reduced echelon form}
\]
\[
\begin{pmatrix}
0 & 4 & 9 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & -7
\end{pmatrix}
\text{ not in echelon form}
\]

(b) Identify the pivots in the following row-echelon matrix. [No explanation is required.]
\[
A = \begin{pmatrix}
-1 & -2 & 7 & 0 & 0 & 3 \\
0 & 0 & 1 & 1 & 5 & -1 \\
0 & 0 & 0 & 2 & -8
\end{pmatrix}
\]

(c) Assume now that the matrix \( A \) from part (b) is the augmented matrix for a system of linear equations.

(i) How many equations and how many variables are there in this system of equations? Explain why.

Answer:
3 Equations (one for each row of \( A \)), and 5 variables (one for each column of \( A \), excluding the last column).

(ii) Which (if any) are free variables? Explain why.

Answer:
\( x_2 \) and \( x_4 \) are free variables, since they correspond to the pivot-free columns in the echelon form of \( A \).
(iii) Give the parametric form of the general solution of this system. From the echelon form of \( A \) we have
\[
\begin{align*}
-x_1 + 2x_2 + 7x_3 &= 3 \\
x_3 + x_4 + 5x_5 &= -1 \\
2x_5 &= -8
\end{align*}
\]
Using reverse substitution, we get
\[
x_5 = -4, \quad x_4 = t, \quad x_3 = -1 - x_4 - 5x_5 = 19 - t, \quad x_2 = s
\]
and
\[
x_1 = -2x_2 + 7x_3 - 3 = -2s + 133 - 7t - 3 = 130 - 2s - 7t.
\]
Thus
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 130 \\ 0 \\ 19 \\ 0 \\ -4 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix},
\]
where \( s, t \in \mathbb{R} \) are arbitrary parameters.

**Problem 2.** [10 points]
Consider the following system of equations:
\[
\begin{align*}
x_1 + 2x_3 &= 5 \\
-2x_2 + 2x_4 &= 1 \\
3x_1 + 6x_2 - x_3 + x_4 &= 11.
\end{align*}
\]
(a) Rewrite this system as a single matrix equation: [No explanation is required.]
**Answer:**
\[
\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 3 & 6 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 11 \end{bmatrix}
\]
(b) Find all the solutions (using whatever form of the equations you most prefer).
**Solution.**
We take the augmented matrix of the system to a row-echelon form:
\[
\begin{bmatrix} 1 & 0 & 2 & 0 & 5 \\ 0 & -2 & 0 & 2 & 1 \\ 3 & 6 & -1 & 1 & 11 \end{bmatrix} \xrightarrow{r_3 \rightarrow r_3 - 3r_1} \begin{bmatrix} 1 & 0 & 2 & 0 & 5 \\ 0 & -2 & 0 & 2 & 1 \\ 0 & -7 & 1 & -4 \end{bmatrix} \xrightarrow{r_3 \rightarrow r_3 + 3r_2} \begin{bmatrix} 1 & 0 & 2 & 0 & 5 \\ 0 & -2 & 0 & 2 & 1 \\ 0 & 0 & -7 & 7 & -1 \end{bmatrix}
\]
Thus in our system $x_4$ is a free variable, and $x_1,x_2,x_3$ are basic variables. We have:

\[
\begin{align*}
  x_1 + 2x_3 &= 5 \\
  -2x_2 + 2x_4 &= 1 \\
  -7x_3 + 7x_4 &= -1.
\end{align*}
\]

Using the reverse substitution method, we have $x_4 = t$, $x_3 = t + \frac{1}{7}$, $x_2 = x_4 - \frac{1}{2} = t - \frac{1}{2}$ and $x_1 = 5 - 2x_3 = -2t + \frac{33}{7}$. Thus

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix} 33 \\ \frac{4}{7} \\ \frac{1}{7} \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix},
\]

where $t \in \mathbb{R}$ is arbitrary.

**Problem 3.** [10 points]

Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

(a) Either show that the columns of $A$ span all of $\mathbb{R}^3$ or explain why they do not.

**Solution.**

We first find a row-echelon form of $A$:

\[
\begin{bmatrix}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9
\end{bmatrix} \rightarrow r_2 \rightarrow r_2 - 2r_1 \\
\rightarrow r_3 \rightarrow r_3 - 3r_1 \\
\begin{bmatrix}
1 & 4 & 7 \\
0 & -3 & -6 \\
0 & -6 & -12
\end{bmatrix} \rightarrow r_3 \rightarrow r_3 - 2r_2 \\
\begin{bmatrix}
1 & 4 & 7 \\
0 & -3 & -6 \\
0 & 0 & 0
\end{bmatrix}
\]

Since not every row of the row-echelon form of $A$ contains a pivot (row 3 does not), Theorem 4 in Ch 1.4 implies that the columns of $A$ do not span $\mathbb{R}^3$.

(b) Either find a nontrivial linear combination of columns of $A$ that adds up to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or explain why it can’t be done.

**Solution.**

We will use the row-echelon form of $A$ to find a nontrivial solution of the linear homogeneous system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$:

\[
\begin{align*}
  x_1 + 4x_2 + 7x_3 &= 0 \\
  -3x_2 - 6x_3 &= 0.
\end{align*}
\]
Then $x_3 = t$, $x_2 = -2t$, $x_1 = -4x_2 - 7x_3 = 8t - 7t = t$. Using $t = 1$ we get a specific nontrivial solution $x_1 = 1$, $x_2 = -2$, $x_3 = 1$.

We now double-check that these values of $x_1, x_2, x_3$ indeed satisfy our requirements:

$$1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$  

Problem 4. [10 points]

Let $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Find all the value(s) of $c$ for which the vector $\begin{bmatrix} 3 \\ -6 \\ c \end{bmatrix}$ is contained in $Span(\vec{v}_1, \vec{v}_2)$.

**Solution.**

The problem is equivalent to finding all $c$ (if any) for which the linear system

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ c \end{bmatrix}$$

is consistent, i.e. has a solution. We find a row-echelon form of the augmented matrix of this system:

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -6 \\ 1 & 2 & c \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -6 \\ 0 & 5/2 & c - 3/2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & c - 3/2 + 15 \end{bmatrix}.$$  

The system is consistent if and only if $c - \frac{3}{2} + 15 = 0$, that is, if and only if $c = -\frac{27}{2}$.

Thus the vector $\begin{bmatrix} 3 \\ -6 \\ c \end{bmatrix}$ is contained in $Span(\vec{v}_1, \vec{v}_2)$ if and only if $c = -\frac{27}{2}$.

Problem 5. [10 points]

For each of the following statements indicate if it is true (T) or false (F). [No explanations need to be given].

(1) Whenever $c \in \mathbb{R}$, $\vec{u}, \vec{v} \in \mathbb{R}^2$ are such that $c\vec{u} = c\vec{v}$ then $\vec{u} = \vec{v}$.

False.

(2) If the echelon form of an $m \times n$ matrix has a pivot in every column then $n \geq m$.

False.
(3) If $A$ is a $2 \times 5$ matrix and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ is a column vector of length 2, then $A\vec{v} \in \mathbb{R}^5$.

(4) If the echelon form of an $m \times n$ matrix $A$ has a pivot in every row, then the matrix equation $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

True.

(5) A system of 7 linear equations in 4 unknowns always has at least one solution.

False.

(6) If $n \geq 1$ then there exists a vector $\vec{v} \in \mathbb{R}^n$ such that $Span(\vec{v}) = \{0\}$.

True. [Take $\vec{v} = \vec{0} \in \mathbb{R}^n$.]

(7) If a system of 7 linear equations in 4 unknowns has at least one solution then it has infinitely many solutions.

False.

(8) A homogeneous system of linear equations is always consistent.

True.

(9) The vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is in $Span(\begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix})$.

False

(10) There does not exist a linear system of equations which has exactly 3 distinct solutions.

True.
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