Problem 1. [10 points]
(a) For each of the following matrices, indicate whether the matrix is in
• echelon form but not reduced echelon form
• reduced echelon form
• not in echelon form
[No explanation is required].
\[
\begin{bmatrix}
1 & 5 & 0 & 9 & 0 \\
0 & 0 & 1 & \pi & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\text{reduced echelon form}
\]
\[
\begin{bmatrix}
5 & -5 & 0 & 7 \\
0 & 0 & \sqrt{7} & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\text{echelon form but not reduced echelon form}
\]
\[
\begin{bmatrix}
0 & 4 & 9 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & -7
\end{bmatrix}
\text{not echelon form}
\]

(b) Identify the pivots in the following row-echelon matrix. [No explanation is required.]
\[
A =
\begin{bmatrix}
1 & -2 & 10 & 0 & 0 & 3 \\
0 & 0 & 1 & 1 & 5 & -1 \\
0 & 0 & 0 & 8 & -16
\end{bmatrix}
\]

(c) Assume now that the matrix \(A\) from part (b) is the augmented matrix for a system of linear equations.
(i) How many equations and how many variables are there in this system of equations? Explain why.

Answer:
3 Equations (one for each row of \(A\)), and 5 variables (one for each column of \(A\), excluding the last column).

(ii) Which (if any) are free variables? Explain why.

Answer:
\(x_2\) and \(x_4\) are free variables, since they correspond to the pivot-free columns in the echelon form of \(A\).

(iii) Give the parametric form of the general solution of this system.
From the echelon form of \(A\) we have
\[
\begin{align*}
  x_1 - 2x_2 & + 10x_3 = 3 \\
  x_3 & + x_4 + 5x_5 = -1 \\
  8x_5 & = -16
\end{align*}
\]
Using reverse substitution, we get
\[
\begin{align*}
x_5 & = -2, \\
x_4 & = t, \\
x_3 & = 9 - x_4 = 9 - t, \\
x_2 & = s \\
x_1 & = 3 + 2x_2 - 10(9 - x_4) = -87 + 2s + 10t.
\end{align*}
\]
Thus
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= \begin{bmatrix}
-87 \\
0 \\
9 \\
0 \\
-2
\end{bmatrix} + s
\begin{bmatrix}
2 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}
+ t
\begin{bmatrix}
10 \\
0 \\
-1 \\
1 \\
0
\end{bmatrix},
\]
where \(s, t \in \mathbb{R}\) are arbitrary parameters.

**Problem 2.** [10 points]
Consider the following system of equations:
\[
\begin{align*}
x_1 + x_3 & = 5 \\
-2x_2 + 2x_4 & = 1 \\
3x_1 + 6x_2 - x_3 + x_4 & = 15.
\end{align*}
\]
(a) Rewrite this system as a single matrix equation: [No explanation is required.]
Answer:
\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & -2 & 0 & 2 \\
3 & 6 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
5 \\
1 \\
15
\end{bmatrix}
\]
(b) Find all the solutions (using whatever form of the equations you most prefer).
Solution.
We take the augmented matrix of the system to a row-echelon form:
\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 5 \\
0 & -2 & 0 & 2 & 1 \\
3 & 6 & -1 & 1 & 15
\end{bmatrix}\xrightarrow{r_3 \rightarrow r_3 - 3r_1}
\begin{bmatrix}
1 & 0 & 1 & 0 & 5 \\
0 & -2 & 0 & 2 & 1 \\
0 & 6 & -4 & 1 & 0
\end{bmatrix}\xrightarrow{r_3 \rightarrow r_3 + 3r_2}
\begin{bmatrix}
1 & 0 & 1 & 0 & 5 \\
0 & -2 & 0 & 2 & 1 \\
0 & 0 & -4 & 7 & 3
\end{bmatrix}
\]
Thus in our system $x_4$ is a free variable, and $x_1, x_2, x_3$ are basic variables. We have:

\[
\begin{cases}
  x_1 + x_3 = 5 \\
  -2x_2 + 2x_4 = 1 \\
  -4x_3 + 7x_4 = 3.
\end{cases}
\]

Using the reverse substitution method, we have $x_4 = t$, $x_3 = -\frac{3}{4} + \frac{7}{4}t$, $x_2 = -\frac{1}{2} + t$ and $x_1 = 5 - x_3 = \frac{23}{4} - \frac{7}{4}t$. Thus

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} = \begin{bmatrix}
  \frac{23}{4} \\
  -\frac{1}{2} \\
  -\frac{3}{4} \\
  0
\end{bmatrix} + t \begin{bmatrix}
  -\frac{7}{4} \\
  1 \\
  \frac{7}{4} \\
  0
\end{bmatrix},
\]

where $t \in \mathbb{R}$ is arbitrary.

**Problem 3.** [10 points]

Let $A = \begin{bmatrix}
  1 & 2 & 3 \\
  4 & 5 & 6 \\
  7 & 8 & 9
\end{bmatrix}$.

(a) Either show that the columns of $A$ span all of $\mathbb{R}^3$ or explain why they do not.

**Solution:**

We fist find a row-echelon form of $A$:

\[
\begin{bmatrix}
  1 & 2 & 3 \\
  4 & 5 & 6 \\
  7 & 8 & 9
\end{bmatrix} \sim \begin{bmatrix}
  1 & 2 & 3 \\
  0 & -3 & -6 \\
  0 & -6 & -12
\end{bmatrix} \sim \begin{bmatrix}
  1 & 2 & 3 \\
  0 & -3 & -6 \\
  0 & 0 & 0
\end{bmatrix}
\]

Since not every row of the row-echelon form of $A$ contains a pivot (row 3 does not), Theorem 4 in Ch 1.4 implies that the columns of $A$ do not span $\mathbb{R}^3$.

(b) Either find a nontrivial linear combination of columns of $A$ that adds up to $\begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}$ or explain why it can’t be done.

**Solution.**

We will use the row-echelon form of $A$ to find a nontrivial solution of the linear homogeneous system $A \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}$.

\[
\begin{cases}
  x_1 + 2x_2 + 3x_3 = 0 \\
  -3x_2 - 6x_3 = 0.
\end{cases}
\]
Then \( x_3 = t \), \( x_2 = -2t \), \( x_1 = -2x_2 - 3x_3 = 4t - 3t = t \). Using \( t = 1 \) we get a specific nontrivial solution \( x_1 = 1 \), \( x_2 = -2 \), \( x_3 = 1 \).

We now double-check that these values of \( x_1, x_2, x_3 \) indeed satisfy our requirements:

\[
1 \cdot \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

**Problem 4.** [10 points]

Let \( \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \).

Find all the value(s) of \( k \) for which the vector \( \begin{bmatrix} 3 \\ 6 \\ k \end{bmatrix} \) is contained in \( \text{Span}(\vec{v}_1, \vec{v}_2) \).

**Solution.**

The problem is equivalent to finding all \( k \) (if any) for which the linear system

\[
\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 6 \\ 1 & 1 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ k \end{bmatrix}
\]

is consistent, i.e. has a solution. We find a row-echelon form of the augmented matrix of this system:

\[
\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 6 \\ 1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 6 \\ 2 & -3 & -9 \\ 0 & 3/2 & k - 3/2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 6 \\ 0 & 3/2 & k - 3/2 - 9 \\ 0 & 0 & k - 3/2 - 9 \end{bmatrix}
\]

The system is consistent if and only if \( k - 3/2 - 9 = 0 \), that is, if and only if \( k = \frac{21}{2} \).

Thus the vector \( \begin{bmatrix} 3 \\ 6 \\ k \end{bmatrix} \) is contained in \( \text{Span}(\vec{v}_1, \vec{v}_2) \) if and only if \( k = \frac{21}{2} \).

**Problem 5.** [10 points]

For each of the following statements indicate if it is true (T) or false (F). [No explanations need to be given].

(1) If \( \vec{u}, \vec{v} \) are vectors in \( \mathbb{R}^8 \) then \( \vec{u} + \vec{v} \) is a vector in \( \mathbb{R}^{16} \).

False

(2) If the echelon form of an \( m \times n \) matrix has a pivot in every column then \( n \geq m \).

False
(3) If $A$ is a $2 \times 5$ matrix and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ is a column vector of length 2, then $A\vec{v} \in \mathbb{R}^5$.

False

(4) If the echelon form of an $m \times n$ matrix $A$ has a pivot in every row, then the matrix equation $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

True

(5) A system of 25 linear equations in 13 unknowns always has at least one solution.

False

(6) If $n \geq 1$ then there exists a vector $\vec{v} \in \mathbb{R}^n$ such that $Span(\vec{v}) = \{\vec{0}\}$.

True.

(7) If a system of 25 linear equations in 13 unknowns has at least one solution then it has infinitely many solutions.

False

(8) A homogeneous system of linear equations is always consistent.

True

(9) The vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in $Span(\begin{bmatrix} 1 \\ 1 \\ 5 \\ 5 \end{bmatrix})$.

False

(10) There does not exist a linear system of equations which has exactly 3 distinct solutions.

True

\footnote{This question was inadvertently omitted from several printed versions of the actual exam, where question (5) was followed by question (7). Those students to whom this happened automatically got a point for part (6) of Problem 5.