Problem 1. [10 points]
(a) For each of the following matrices, indicate whether the matrix is in:
- echelon form but not reduced echelon form
- reduced echelon form
- not in echelon form

[No explanation is required.]

\[ \begin{bmatrix} 1 & 5 & 0 & 9 & 0 \\ 0 & 0 & 1 & \pi & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ reduced echelon form} \]

\[ \begin{bmatrix} 5 & -5 & 0 & 7 \\ 0 & 0 & \sqrt{7} & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ echelon form but not reduced echelon form} \]

\[ \begin{bmatrix} 0 & 4 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -7 \end{bmatrix} \text{ not echelon form} \]

(b) Identify the pivots in the following row-echelon matrix. [No explanation is required.]

\[ A = \begin{bmatrix} 1 & -2 & 10 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 8 & -16 \end{bmatrix} \]

(c) Assume now that the matrix \( A \) from part (b) is the augmented matrix for a system of linear equations.

(i) How many equations and how many variables are there in this system of equations? Explain why.

Answer: 3 Equations (one for each row of \( A \)), and 5 variables (one for each column of \( A \), excluding the last column).

(ii) Which (if any) are free variables? Explain why.

Answer:
$x_2$ and $x_4$ are free variables, since they correspond to the pivot-free columns in the echelon form of $A$.

(iii) Give the parametric form of the general solution of this system. From the echelon form of $A$ we have

\[
\begin{align*}
     x_1 - 2x_2 + 10x_3 &= 3 \\
     3x_3 + x_4 + 5x_5 &= -1 \\
     8x_5 &= -16
\end{align*}
\]

Using reverse substitution, we get $x_5 = -2$, $x_4 = t$, $x_3 = 9 - x_4 = 9 - t$, $x_2 = s$ and $x_1 = 3 + 2x_2 - 10(9 - x_4) = -87 + 2s + 10t$.

Thus

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
    x_5
\end{bmatrix} =
\begin{bmatrix}
    -87 \\
    0 \\
    9 \\
    0 \\
   -2
\end{bmatrix} + s
\begin{bmatrix}
   2 \\
   1 \\
   0 \\
   0 \\
   0
\end{bmatrix} + t
\begin{bmatrix}
   10 \\
   0 \\
   0 \\
   0 \\
   0
\end{bmatrix},
\]

where $s, t \in \mathbb{R}$ are arbitrary parameters.

**Problem 2.** [10 points]

Consider the following system of equations:

\[
\begin{align*}
    x_1 + x_3 &= 5 \\
   -2x_2 + 2x_4 &= 1 \\
   3x_1 + 6x_2 - x_3 + x_4 &= 15.
\end{align*}
\]

(a) Rewrite this system as a single matrix equation: [No explanation is required.]

**Answer:**

\[
\begin{bmatrix}
   1 & 0 & 1 & 0 \\
   0 & -2 & 0 & 2 \\
   3 & 6 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} =
\begin{bmatrix}
   5 \\
   1 \\
   15
\end{bmatrix}
\]

(b) Find all the solutions (using whatever form of the equations you most prefer).

**Solution.**

We take the augmented matrix of the system to a row-echelon form:

\[
\begin{bmatrix}
   1 & 0 & 1 & 0 & 5 \\
   0 & -2 & 0 & 2 & 1 \\
   3 & 6 & -1 & 1 & 15
\end{bmatrix}
\overset{r_3 \rightarrow r_3 - 3r_1}{\sim}
\begin{bmatrix}
   1 & 0 & 1 & 0 & 5 \\
   0 & -2 & 0 & 2 & 1 \\
   0 & 6 & -4 & 1 & 0
\end{bmatrix}
\overset{r_3 \rightarrow r_3 + 3r_2}{\sim}
\begin{bmatrix}
   1 & 0 & 1 & 0 & 5 \\
   0 & -2 & 0 & 2 & 1 \\
   0 & 0 & -4 & 7 & 3
\end{bmatrix}
\]
Thus in our system \( x_4 \) is a free variable, and \( x_1, x_2, x_3 \) are basic variables. We have:

\[
\begin{align*}
  x_1 + x_3 &= 5 \\
  -2x_2 + 2x_4 &= 1 \\
  -4x_3 + 7x_4 &= 3.
\end{align*}
\]

Using the reverse substitution method, we have \( x_4 = t \), \( x_3 = -\frac{3}{4} + \frac{7}{4}t \), \( x_2 = -\frac{1}{2} + t \) and \( x_1 = 5 - x_3 = \frac{23}{4} - \frac{7}{4}t \). Thus

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} = \begin{bmatrix}
  23/4 \\
  -1/2 \\
  -3/4 \\
  0
\end{bmatrix} + t \begin{bmatrix}
  -7/4 \\
  1/4 \\
  1/4 \\
  1
\end{bmatrix},
\]

where \( t \in \mathbb{R} \) is arbitrary.

**Problem 3.** [10 points]

Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \).

(a) Either show that the columns of \( A \) span all of \( \mathbb{R}^3 \) or explain why they do not.

**Solution:**

We first find a row-echelon form of \( A \):

\[
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Since not every row of the row-echelon form of \( A \) contains a pivot (row 3 does not), Theorem 4 in Ch 1.4 implies that the columns of \( A \) do not span \( \mathbb{R}^3 \).

(b) Either find a nontrivial linear combination of columns of \( A \) that adds up to \( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) or explain why it can’t be done.

**Solution.**

We will use the row-echelon form of \( A \) to find a nontrivial solution of the linear homogeneous system \( A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \).

\[
\begin{align*}
  x_1 + 2x_2 + 3x_3 &= 0 \\
  -3x_2 - 6x_3 &= 0.
\end{align*}
\]
Then \( x_3 = t, x_2 = -2t, x_1 = -2x_2 - 3x_3 = 4t - 3t = t. \) Using \( t = 1 \) we get a specific nontrivial solution \( x_1 = 1, x_2 = -2, x_3 = 1. \)

We now double-check that these values of \( x_1, x_2, x_3 \) indeed satisfy our requirements:

\[
1 \cdot \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

**Problem 4.** [10 points]

Let \( \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}. \)

Find all the value(s) of \( k \) for which the vector \( \begin{bmatrix} 3 \\ 6 \\ k \end{bmatrix} \) is contained in \( \text{Span}(\vec{v}_1, \vec{v}_2) \).

**Solution.**

The problem is equivalent to finding all \( k \) (if any) for which the linear system

\[
\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 6 \\ 1 & 1 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ k \end{bmatrix}
\]

is consistent, i.e. has a solution. We find a row-echelon form of the augmented matrix of this system:

\[
\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 6 \\ 1 & 1 & k \end{bmatrix} \xrightarrow{r_3 \rightarrow r_3 - \frac{3}{2} r_1} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 6 \\ 0 & \frac{3}{2} & k - \frac{3}{2} \end{bmatrix} \xrightarrow{r_3 \rightarrow r_3 - \frac{3}{2} r_2} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & k - \frac{3}{2} - 9 \end{bmatrix}
\]

The system is consistent if and only if \( k - \frac{3}{2} - 9 = 0 \), that is, if and only if \( k = \frac{21}{2} \).

Thus the vector \( \begin{bmatrix} 3 \\ 6 \\ k \end{bmatrix} \) is contained in \( \text{Span}(\vec{v}_1, \vec{v}_2) \) if and only if \( k = \frac{21}{2} \).

**Problem 5.** [10 points]

For each of the following statements indicate if it is true (T) or false (F).

[No explanations need to be given].

(1) If \( \vec{u}, \vec{v} \) are vectors in \( \mathbb{R}^8 \) then \( \vec{u} + \vec{v} \) is a vector in \( \mathbb{R}^{16} \).

False

(2) If the echelon form of an \( m \times n \) matrix has a pivot in every column then \( n \geq m \).

False
(3) If \( A \) is a \( 2 \times 5 \) matrix and \( \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \) is a column vector of length 2, then \( A\vec{v} \in \mathbb{R}^5 \).

False

(4) If the echelon form of an \( m \times n \) matrix \( A \) has a pivot in every row, then the matrix equation \( A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{b} \) is consistent for every \( \vec{b} \in \mathbb{R}^m \).

True

(5) A system of 25 linear equations in 13 unknowns always has at least one solution.

False

(6) If \( n \geq 1 \) then there exists a vector \( \vec{v} \in \mathbb{R}^n \) such that \( \text{Span}(\vec{v}) = \{ \vec{0} \} \).

True

(7) If a system of 25 linear equations in 13 unknowns has at least one solution then it has infinitely many solutions.

True

(8) A homogeneous system of linear equations is always consistent.

True

(9) The vector \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) is in \( \text{Span}(\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix}) \).

False

(10) There does not exist a linear system of equations which has exactly 3 distinct solutions.

True

\(^1\)This question was inadvertently omitted from several printed versions of the actual exam, where question (5) was followed by question (7). Those students to whom this happened automatically got a point for part (6) of Problem 5.