Math 213, Quiz 7 (Solution); Friday, March 7, 2008

1. (a) Find the general solution of the following recurrence relation:
\[ a_n = -6a_{n-1} - 9a_{n-2}. \]

(b) Find a sequence \( a_n \) such that
\[
\begin{align*}
    a_n &= -6a_{n-1} - 9a_{n-2}, \quad n \geq 2 \\
    a_0 &= 1, a_1 = 0.
\end{align*}
\]

Provide a detailed explanation of your answers.

Solution.

(a) The characteristic equation of this recurrence relation is:
\[ r^2 + 6r + 9 = 0 \]
\[ (r + 3)^2 = 0. \]

Hence the general solution of the recurrence relation \( a_n = -6a_{n-1} - 9a_{n-2} \) is:
\[ a_n = (\alpha + \beta n)(-3)^n = \alpha(-3)^n + \beta n(-3)^n, \]
where \( \alpha, \beta \in \mathbb{R} \) are arbitrary constants.

(b) Using the result of (a), we know that \( a_n \) has the form \( a_n = \alpha(-3)^n + \beta n(-3)^n \) for some \( \alpha, \beta \in \mathbb{R} \). To find the specific values of \( \alpha \) and \( \beta \), we use the initial conditions \( a_0 = 1, a_1 = 0 \) that yield:
\[
\begin{align*}
    \alpha(-3)^0 + \beta 0 (-3)^0 &= 1 \\
    \alpha(-3)^1 + \beta n(-3)^1 &= 0
\end{align*}
\]
\[ \Rightarrow \begin{cases} 
    \alpha = 1 \\
    -3\alpha - 3\beta = 0,
\end{cases} \quad \Rightarrow \begin{cases} 
    \alpha = 1 \\
    \beta = -1.
\end{cases} \]

Thus \( a_n = (-3)^n - n(-3)^n. \)