Math 285 - Spring 2012 - Exam 1 Solutions

**Problem 1** (4 points). Show the initial value problem \( xy' = y - 5 \), \( y(0) = 5 \) has infinitely many solutions. Does this contradict the Existence and Uniqueness Theorem for 1st order differential equations of the form \( y' = f(x,y) \)? Explain.

**Solution:** This is a separable equation. Write it as

\[
\frac{dy}{y - 5} = \frac{dx}{x}
\]

Thus integrating both sides we obtain

\[
\ln(y - 5) = \ln x + C
\]

Therefore, the general solution is \( y - 5 = kx \) where \( k \) is a constant. Note that the general solution satisfies the initial condition \( y(0) = 5 \) for all \( k \), so there are infinitely many solutions to the IVP. This does not contradict the EUT as \( f(x,y) = \frac{y - 5}{x} \) is not continuous at \((0,5)\).

**Problem 2** (4 points). Solve \((x^2 + 1)y' = x(y - 1)\) as a linear first order differential equation. Find the particular solution with the initial value \( y(0) = 2 \).

**Answer:** Divide by \( x^2 + 1 \) and write the equation as

\[
y' - \frac{x}{x^2 + 1}y = -\frac{x}{x^2 + 1}
\]

\((*)\)

The integrating factor is

\[
\rho(x) = e^{\int -\frac{x}{x^2 + 1} \, dx} = e^{-\frac{1}{2} \ln(x^2 + 1)} = (x^2 + 1)^{-\frac{1}{2}}
\]

Multiply both sides of \((*)\) by \( \rho(x) \) and write the equation as

\[
\frac{d}{dx}(\rho y) = -\frac{x}{x^2 + 1} \cdot \rho = -x(x^2 + 1)^{-\frac{3}{2}}
\]

Hence

\[
\rho y = \int -x(x^2 + 1)^{-\frac{3}{2}} \, dx = (x^2 + 1)^{-\frac{1}{2}} + C
\]

So, divide by \( \rho \) and we obtain

\[
y = 1 + C(x^2 + 1)^{\frac{1}{2}}
\]

Note that \( y(0) = 1 + C \), thus \( C = 1 \) in the particular solution satisfying the initial value \( y(0) = 2 \),

\[
y = 1 + (x^2 + 1)^{\frac{1}{2}}
\]
**Problem 3** (3 points). Solve $xy' = x^2 + y^2$.

**Answer:** Write the equation as

$$y' = \frac{x^2 + y^2}{xy} = \frac{x}{y} + \frac{y}{x}$$

This is a homogeneous equation, so let $u = y/x$. Then substituting in the DE we get

$$y' = u + xu' = \frac{1}{u} + u$$

Canceling $u$ on both sides we have $xu' = \frac{1}{u}$, which can be solved as a separable equation. Write is as

$$udu = \frac{dx}{x}$$

Therefore,

$$\frac{1}{2}u^2 = \ln|x| + C$$

Substitute back $u = y/x$ and obtain the general solution as

$$y^2 = 2x^2(\ln|x| + C)$$

**Problem 4** (4 points). Determine if the equation $(x + ye^{xy})dx + (y + xe^{xy})dy = 0$ is exact. If yes, find a particular solution with the initial value $y(0) = 2$.

**Answer:** Write the equation as $Mdx + Ndy = 0$, where $M = x + ye^{xy}$ and $N = y + xe^{xy}$. The exactness criterion requires that $M_y = N_x$. We compute

$$M_y = e^{xy} + xye^{xy} = N_x$$

Thus the condition is satisfied. To solve the equation, we need to find a function $F(x, y)$ such that $F_x = x + ye^{xy}$ and $F_y = y + xe^{xy}$. Integrating both sides of the first equation with respect to $x$ we obtain

$$F = \frac{1}{2}x^2 + e^{xy} + h(y)$$

where $h$ is a function purely in $y$. Thus

$$F_y = ye^{xy} + h'(y)$$

Comparing with $F_y = y + xe^{xy}$ with obtain $h'(y) = y$, so $h(y) = \frac{1}{2}y^2$. Thus

$$F(x, y) = \frac{1}{2}x^2 + e^{xy} + \frac{1}{2}y^2$$

Therefore, the general solution is of the form $F(x, y) = C$, or $\frac{1}{2}x^2 + e^{xy} + \frac{1}{2}y^2 = C$, where $C$ is a constant. Since $y(0) = 2$, let $x = 0$ and $y = 2$ in the general solution and we obtain $C = 3$. Thus the particular solution is

$$\frac{1}{2}x^2 + e^{xy} + \frac{1}{2}y^2 = 3$$

or $x^2 + 2e^{xy} + y^2 = 6$
Problem 5 (3 points). Solve the initial value problem $y'' = 2yy', y(0) = 0, y'(0) = 1$.

Answer: This is a reducible 2nd order DE missing x. Substitute $u = y'$ as a function of $y$ and $y'' = u \cdot \frac{du}{dy}$. Then the DE becomes

$$u \cdot \frac{du}{dy} = 2yu$$

which is a separable equation. After canceling $u$ on the both sides, we can write it as $du = 2ydy$ which gives the general solution $u = y^2 + C$. Substitute back $y'$ for $u$,

$$y' = y^2 + C$$

The initial values $y(0) = 0, y'(0) = 1$ imply that $C = 1$, so we obtain $y' = y^2 + 1$. This DE can be solved as separable equation:

$$\frac{dy}{y^2 + 1} = dx \implies \tan^{-1}(y) = x + C \implies y = \tan(x + C)$$

Note that $C = 0$ since $y(0) = 0$. Thus the particular solution is $y = \tan x$.

Problem 6 (2 points). Consider a population model in which the death rate and birth rate per unit of population are $\beta(t) = 2 + P(t)$ and $\delta(t) = 3$ respectively. If $P(0) = 2$, then show that there will be a Doomsday!

Answer: The DE governing the population function $P(t)$ is

$$P' = (\beta - \delta)P = (-1 + P)P = -(1 - P)P$$

which is a separable equation. Thus

$$\frac{dP}{(1 - P)P} = -dt \implies \ln\left(\frac{P}{1 - P}\right) = -t + C$$

Then

$$\frac{P}{1 - P} = ke^{-t}$$

Where $k = \frac{P(0)}{1 - P(0)} = -2$. Thus

$$P(t) = \frac{1}{1 - \frac{1}{2}e^t}.$$  

Note that the denominator is zero when $t = \ln 2 > 0$. Hence $P(t)$ tends to $\infty$ as $t \to \ln 2$, the doomsday!