BOUNDS ON THE NUMBER OF NON-SIMPLE GEODESICS ON A SURFACE

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1. Introduction

Let $S$ be a genus $g$ surface with $n$ boundary components, and let $X$ be a negatively curved metric on $S$. Closed geodesics on surfaces have been studied extensively over the years. In this paper, we give upper and lower bounds on the number of closed geodesics on $S$ given upper bounds on length and self-intersection number. The lower bound follows from a lower bound on the number of closed geodesics in a pair of pants, and is proven in a previous paper. The upper bound comes from looking at closed geodesics on a closed surface with a flat metric.

1.1. Previous results on an arbitrary surface. There was a lot of work done in the 70’s and 80’s on counting closed geodesics on a negatively curved surface $S$ from the point of view of counting closed orbits of the geodesic flow. Let $G^c$ be the set of closed geodesics on $S$ and let

$$G^c(L) = \{ \gamma \in G^c \mid l(\gamma) \leq L \}$$

where $l(\gamma)$ is the length of $\gamma$. The famous result in Margulis’s thesis states that if $S$ is negatively curved with a complete, finite volume metric, then

$$\#G^c(L) \sim e^{\frac{\delta L}{L}}$$

where $\delta$ is the topological entropy of the geodesic flow, and where $f(L) \sim g(L)$ if $\lim_{L \to \infty} \frac{f(L)}{g(L)} = 1$ [Mar70]. (Note that $\delta = 1$ when $S$ is hyperbolic.) A version of this result for hyperbolic surfaces was first proven by Huber [Hub59]. There are also many other, later versions of this result for non-closed surfaces. For example, see [CdV85, Pat88, LP82, Lal89] and [Gui86].

Recently, there has been work on the dependence of the number of closed geodesics on their self-intersection number as well as length. If $i(\gamma, \gamma)$ denotes the transverse self-intersection number of $\gamma \in G^c$, let

$$G^c(L, K) = \{ \gamma \in G^c \mid l(\gamma) \leq L, i(\gamma, \gamma) \leq K \}$$

Then we can pose the following question.

Question 1. If $K = K(L)$ is a function of $L$, what is the asymptotic growth of $\#G^c(L, K)$ in terms of $L$?

As part of her thesis, Mirzakhani showed that for a hyperbolic surface $S$ of genus $g$ with $n$ punctures,

$$\#G^c(L, 0) \sim c(S)L^{6g-6+2n}$$
where $c(S)$ is a constant depending only on the geometry of $S$ [Mir08]. Rivin extended this result to geodesics with at most one self-intersection, to get that
\[
\#\mathcal{G}^c(L,1) \sim c'(S)L^{6g-6+2n}
\]
where $c'(S)$ is another constant depending only on the geometry of $S$ [Riv12].

For arbitrary functions $K = K(L)$, no asymptotic bounds are yet known. We can instead ask the following question as a first step to finding asymptotics.

**Question 2.** Given arbitrary $L$ and $K$, what are the best upper and lower bounds we can get on $\#\mathcal{G}^c(L,K)$?

Trivial bounds come from the fact that $\#\mathcal{G}^c(L,0) \leq \#\mathcal{G}^c(L,K) \leq \#\mathcal{G}^c(L)$, but these bounds do not have any dependence on $K$. A first bound that does depend on $K$ can be achieved as follows. The mapping class group, $\text{Mod}_{g,n}$, acts on the set of closed geodesics and preserves self-intersection number. So consider the set of $\text{Mod}_{g,n}$ orbits of geodesics. Let $f(K)$ be the number of $\text{Mod}_{g,n}$ orbits of closed geodesics with at most $K$ self intersections. Just as there are finitely many topological types of simple closed curves, the number $f(K)$ is finite. Then by combining the asymptotic result in [ABEM12] with results in [Bas13, Ker80, Wol79], we can get that for large $L$ and arbitrary $K$,
\[
\#\mathcal{G}^c(L,K) \asymp f(K)L^{6g-6+2n}
\]
where $f(K)$ is the (finite) number of mapping class group orbits in $\mathcal{G}^c(L,K)$. We write $A \asymp B$ if there is a constant $c$ depending only on the geometry of $S$ s.t. $\frac{1}{c}B \leq A \leq cB$. However, $f(K)$ is not known for most $K$. Our goal is to get bounds that are explicit in both length $L$ and intersection number $K$.

1.2. **What we show.** In this paper, we prove the following upper bound on $\#\mathcal{G}^c(L,K)$.

**Theorem 1.1.** Let $X$ be a negatively curved metric on a genus $g$ surface $S$ with $n$ geodesic boundary components. For any $L$ and any $K \geq 1$, we bound the size of $\mathcal{G}^c(L,K)$ on $X$ from above as follows:
\[
\#\mathcal{G}^c(L,K) \leq \min \left\{ c_X e^{\delta L} \left( e^{L} \right)^{\sqrt{K}} \right\}
\]
where $c_X$ depends on the geometry of $X$, $\delta$ is the topological entropy of the geodesic flow on $S$, and $e^{\sqrt{K}}$ depends only on the topology of $S$.

This theorem gives us better bounds than Margulis’s bound on $\#\mathcal{G}^c(L)$ when $K = o\left( \frac{L^2}{\ln^2 L} \right)$:

**Corollary 1.2.** If $K = K(L)$ is a function of $L$ such that $K = o\left( \frac{L^2}{\ln^2 L} \right)$, then for any $0 < c < 1$,
\[
\frac{\#\mathcal{G}^c(L,K)}{\#\mathcal{G}^c(L)} < e^{-cL}
\]
for all $L$ large enough, depending on $c$ and $X$.

The following lower bound on $\#\mathcal{G}^c(L,K)$ is proven in a previous paper:
Theorem 1.3. Let $S$ be a genus $g$ surface with $n$ geodesic boundary components, and let $X$ be a negatively curved metric on $S$. Then whenever $K > 12$ and $L > 3s_X \sqrt{K}$ we have

$$\#G^c(L, K) \geq c_X \left( \frac{L}{3\sqrt{K}} - s_X \right)^{6g-6+2n} 2^{\frac{2K}{3X}}$$

where $s_X$ and $c_X$ are constants that depend only on the metric $X$.

As $L$ goes to infinity, this theorem suggests a way to interpolate between the case when $K$ is a constant and the case when $K$ grows like $L^2$. If $K$ is a constant, and $L$ is very large, this theorem says $\#G(L, K) \geq c'_X L^{6g-6+2n}$, for $c'_X$ a new constant depending on $X$. This is consistent with the asymptotic results in [Mir08, Riv12] when $K = 0$ and 1. For $K = O(L^2)$, however, we have that $\frac{L}{3\sqrt{K}} = O(1)$, and Theorem 1.3 gives an exponential lower bound on $\#G^c(L, K)$ in $L$ that is consistent with Margulis’s result [Mar70]. This theorem demonstrates the transition from polynomial to exponential growth of the number of geodesics on $S$ in terms of their length and self-intersection number.
2. Upper bound

We actually show the upper bound on \( \#G^c(L, K) \) for closed surfaces. An upper bound for surfaces with boundary follows as a corollary.

**Theorem 2.1.** Let \( X_{-1} \) be a negatively curved metric on a closed genus \( g \) surface \( S \). For any \( L \) and any \( K \geq 1 \), we bound the size of \( G^c(L, K) \) on \( X_{-1} \) from above as follows:

\[
\#G^c(L, K) \leq \min \left\{ c_X e^{\delta L} \sqrt{c_S} \right\}
\]

where \( c_X \) depends on the geometry of \( X_{-1} \), \( c_S \) depends only on the topology of \( S \) and \( \delta \) is the topological entropy of the geodesic flow.

Note that by Margulis’s theorem [Mar70], \( \#G^c(L, K) \leq c_X e^{\delta L} \) for some constant \( c_X \). We just need to prove that \( \#G^c(L, K) \leq (c_X L)^{c_S} \sqrt{c_S} \).

A corollary of this theorem is an upper bound for \( \#G^c(L, K) \) on a surface with boundary:

**Corollary 2.2.** Let \( X_{-1} \) be a negatively curved metric on a genus \( g \) surface \( S \) with \( n \) geodesic boundary components. For any \( L \) and any \( K \geq 1 \), we bound the size of \( G^c(L, K) \) on \( X_{-1} \) from above as follows:

\[
\#G^c(L, K) \leq \min \left\{ c_X e^{\delta L} \sqrt{c_S} \right\}
\]

where \( c_X \) depends on the geometry of \( X_{-1} \), \( \delta \) is the topological entropy of the geodesic flow on \( S \), and \( c_S \) depends only on the topology of \( S \).

**Proof.** Let \( X_{-1} \) be a negatively curved metric on a genus \( g \) surface \( S \) with \( n \) geodesic boundary components. We can double it along its boundary to get a closed genus \( 2g + n - 1 \) surface \( S' \) with metric \( X'_{-1} \). \( S \) injects into \( S' \) in a canonical way, so that \( X'_{-1} \) pulls back to the metric \( X_{-1} \) on \( S \). Thus, closed geodesics in \((S', X'_{-1})\) either pull back to closed geodesics or multi-arcs in \((S, X_{-1})\).

By Theorem 2.1, \( \#G^c(L, K) \) is at most \( (c_X L)^{c_S} \sqrt{c_S} \) on \((S, X'_{-1})\), where \( g' = 2g + n - 1 \), and where \( c_X' \) is a constant depending on \( X'_{-1} \), and therefore on \( X_{-1} \). Therefore, on \( X_{-1} \),

\[
\#G^c(L, K) \leq (c_X L)^{c_S} \sqrt{c_S}
\]

where we set \( c_X = c_X' \) and \( c_S = c_S' \).

Furthermore, by extensions of the theorem of Margulis to surfaces with boundary due, for example, to Guillopé [Gui94], \( \#G^c(L) \) on \((S, X_{-1})\) is asymptotically \( \frac{c_X L}{\delta L} \), where \( \delta \) is the topological entropy of the geodesic flow on \( S \). Thus, by adjusting the constant \( c_X \), we get that \( G^c(L, K) \leq c_X e^{\delta L} \), as well. This gives us the corollary. \( \square \)

3. Reduction to flat surfaces

Theorem 2.1 follows from counting geodesics in a flat metric. This is by the following lemma:

**Lemma 3.1.** Let \( S \) be a surface. Let \( X \) be a negatively curved metric on \( S \) and let \( X_0 \) be a flat metric. Then there is a constant \( \lambda \) depending on \( X \) and \( X_0 \) so that for all closed geodesics \( \gamma \in G^c \),

\[
\frac{1}{\lambda} l_0(\gamma) \leq l(\gamma) \leq \lambda l_0(\gamma)
\]


where \( l(\gamma) \) is the length of the geodesic representative of \( \gamma \) in \( X \) and \( l_0(\gamma) \) is the length of the geodesic representative of \( \gamma \) in \( X_0 \).

**Proof.** We will use the set \( \mathcal{C}(S) \) of geodesic currents for this proof. This is the set of Borel, geodesic-flow invariant measures on \( T_1(S) \). The set of closed geodesics \( \mathcal{G}^c \) embeds in \( \mathcal{C}(S) \). Also, there is a well-defined intersection number \( i(\cdot, \cdot) \) on pairs of geodesics currents that restricts to the usual geometric intersection number on \( \mathcal{G}^c \times \mathcal{G}^c \). This intersection number is continuous and bi-linear. As geodesic currents do not appear anywhere else in this paper, we refer the reader to [Bon88] for more details.

Given the negatively curved metric \( X \), we can define the associated Liouville current \( \mu \in \mathcal{C}(S) \). This geodesic current has the property that for each closed geodesic \( \gamma \in \mathcal{G}^c \),

\[
i(\gamma, \mu) = l(\gamma).
\]

By [DLR10, Theorem 4], each flat metric \( X_0 \) on \( S \) can also be represented by a geodesic current \( \mu_0 \in \mathcal{C}(S) \). They show that these geodesic currents behave just like the ones for negatively curved metrics. That is, for each \( \gamma \in \mathcal{G}^c \), we have

\[
i(\gamma, \mu_0) = l_0(\gamma).
\]

Consider the function

\[
f : \mathcal{C}(S) \to \mathbb{R} : \gamma \mapsto \frac{i(\gamma, \mu)}{i(\gamma, \mu_0)}
\]

This map has the property that \( f(c \cdot \gamma) = f(\gamma) \) so it descends to a map \( f : \mathbb{P}\mathcal{C}(S) \to \mathbb{R} \), where \( \mathbb{P}\mathcal{C}(S) \) is the set of projectivized geodesic currents. By [Bon88, Corollary 5], \( \mathbb{P}\mathcal{C}(S) \) is compact. As \( l(\gamma) \) and \( l_0(\gamma) \) are never 0, \( f \) is a continuous, positive function on a compact set. Therefore, there are constants \( c_1, c_2 > 0 \) so that

\[
c_1l_0(\gamma) \leq l(\gamma) \leq c_2l_0(\gamma)
\]

for each closed geodesic \( \gamma \in \mathcal{G}^c \). \( \square \)

Let \( \mathcal{G}^c_Y(L, K) \) denote the set \( \mathcal{G}^c(L, K) \) for a metric \( Y \) on \( S \). Let \( X_{-1} \) be a negatively curved metric on \( S \) and let \( X_0 \) be a flat metric. Then Lemma 3.1 implies that

\[
\mathcal{G}^c_{X_{-1}}(L, K) \subset \mathcal{G}^c_{X_0}(\frac{1}{\lambda}L, K)
\]

for each \( L, K > 0 \). Therefore,

\[
\#\mathcal{G}^c_{X_{-1}}(L, K) \leq \#\mathcal{G}^c_{X_0}(\frac{1}{\lambda}L, K)
\]

We give an upper bound on \( \#\mathcal{G}^c_{X_0}(L, K) \) in Theorem 4.1. This upper bound directly implies Theorem 4.1.

4. **Bounding the number of closed geodesics in a flat metric**

Let \( X_0 \) be a flat metric on \( S \) with one singular point, which we denote \( s \). We wish to count closed geodesics on \( X_0 \) that pass through \( s \). If a geodesic \( \gamma \) on \( X_0 \) does not pass through \( s \), then it is contained in a flat cylinder. But then \( \gamma \) must be simple. We know how to count simple closed geodesics, so counting geodesics through \( s \) will allow us to count all geodesics on \( X_0 \).

Let \( \mathcal{G}^c_s \) denote the set of closed geodesics on \( X_0 \) that are not contained in any cylinder. Then let

\[
\mathcal{G}^c_s(L) = \{ \gamma \in \mathcal{G}^c_s | l_0(\gamma) \leq L \}
\]
and
\[ G^c_c(L, K) = \{ \gamma \in G^c_c \mid l_0(\gamma) \leq L, i(\gamma, \gamma) \leq K \} \]
Here, \( l_0(\gamma) \) denotes the geodesic length of \( \gamma \) on \( X_0 \) and \( i(\gamma, \gamma) \) denotes the least transverse self-intersection number of all closed curves in the free homotopy class of \( \gamma \).

**Theorem 4.1.** Let \( X_0 \) be a flat metric with one singular point on \( S \). For any \( L \) and any \( K \geq 1 \), we bound the size of \( G^c_c(L, K) \) on \( X_0 \) from above as follows:
\[ \#G^c_c(L, K) \leq (c_s L)^{c_s \sqrt{K}} \]
where \( c_s \) depends on the geometry of \( X_0 \), and \( c_g \) depends only on the topology of \( S \).

**Corollary 4.2.** For all \( L \) and all \( K \geq 1 \), we bound \( \#G^c(L, K) \) on \( X_0 \) by
\[ \#G^c(L, K) \leq (c_0 L)^{c_s \sqrt{K}} \]
where \( c_0 \) depends on the geometry of \( X_0 \), and \( c_g \) depends only on the topology of \( S \).

**Proof.** We know that
\[ G^c(L, K) \setminus G^c_c(L, K) = \{ \gamma \in G^c \mid \gamma \text{ lies in a cylinder} \} \]
By [Mas90], there is some universal constant \( c_{cyl} \) so that the number of cylinders that contain a closed geodesic of length at most \( L \) is at most \( c_{cyl} L^2 \). Thus,
\[ \#G^c(L, K) \leq \#G^c_c(L, K) + c_{cyl} L^2 \]
Since \( \#G^c_c(L, K) \leq (c_s L)^{c_s \sqrt{K}} \), and since there is some \( L_0 \) so that \( G^c(L_0, K) = \emptyset \) for all \( K \), there exists a constant \( c_X \) depending only on \( X \) so that
\[ \#G^c(L, K) \leq (c_X L)^{c_s \sqrt{K}} \]
\[ \square \]

5. **Strategy of the proof**

Let \( C \) be the set of saddle connections on \( X_0 \). Because there is just one singular point, we can think of each \( \sigma \in C \) as both a simple arc \( \sigma : s \mapsto s \) and as a simple closed geodesic \( \bar{\sigma} \). Since no geodesic in \( G^c_c \) lies in a flat cylinder, each \( \gamma \in G^c_c \) can be written as the concatenation of saddle connections:
\[ \gamma = \sigma_1 \ldots \sigma_n \text{ with } \sigma_i \in C, \forall i \]
This should be thought of as a decomposition of \( \gamma \) into simple closed curves. Note that the sequence \( \sigma_1, \ldots, \sigma_n \) uniquely determines \( \gamma \).

This is why we work with the flat metric \( X_0 \) instead of working directly with a negatively curved metric \( X \). If, instead, \( \gamma \) were a geodesic in a hyperbolic metric, there is also a way to write it in terms of simple closed curves. However, simple closed geodesics in a hyperbolic metric do not meet at a single point. So a map from \( \gamma \) to a sequence of simple, closed, hyperbolic geodesics \( \sigma_1, \ldots, \sigma_n \) is not injective: \( \sigma_1, \ldots, \sigma_n \) do not uniquely determine \( \gamma \) in this case. One has to specify arcs to connect \( \sigma_i \) to \( \sigma_{i+1} \), for each \( i \). This leads to complications which are circumvented by choosing \( \gamma \) to be a geodesic with respect to a flat metric.

One approach to counting geodesics in \( G^c_c(L, K) \) is as follows. Suppose we can find a function \( N(L, K) \) so that if \( \gamma = \sigma_1 \ldots \sigma_n \in G^c_c(L, K) \) for \( \sigma_i \in C, \forall i \), then
\[ n \leq N(L, K) \]
If $l_0(\gamma) \leq L$, then $l_0(\sigma_i) \leq L, \forall i$. The number of saddle connections of length at most $L$ grows like $O(L^2)$. Thus,

$$\#G^c_*(L, K) \leq cL^{2N(L,K)}$$

for some constant $c$.

The problem with this approach is that even simple closed geodesics of length $L$ can be written as roughly length $L$ sequences of saddle connections. So the best we could do is $N(L, K) \approx L$, giving us a bound of $\#G^c_*(L, K) \lesssim L^L$. This is not very good. But we get over this problem by replacing sequences of saddle connections with sequences of simple arcs. In particular, the proof goes as follows.

1. We first define what we mean by a simple geodesic arc $\delta: s \mapsto s$ (Definition 7.1.) This is complicated by the fact that geodesic arcs in a flat metric can follow the same saddle connection multiple times, and are therefore not self-transverse.

2. Let $C_0 = \{ \delta: s \mapsto s \mid \delta \text{ simple geodesic arc} \}$
   and $C_0(L) = \{ \delta \in C_0 \mid l_0(\delta) \leq L \}$

   We bound the size of $C_0(L)$:
   $$\#C_0(L) \lesssim O(L^{c_g})$$
   where $c_g$ is a constant depending only on the genus of $S$ (Lemma 8.1).

3. Lastly, we find a constant $N(L, K)$ so that if $\gamma = \delta_1 \ldots \delta_n \in G^c_*(L, K)$, with $\delta_i \in C_0, \forall i$, then
   $$n \leq N(L, K)$$

   In fact, $N(L, K) \lesssim \min \{ \sqrt{K}, L \}$

   (See Lemma 9.1 for the precise statement.)

   The fact that a geodesic of length $L$ can be decomposed into a most $L$ simple geodesic arcs is not so surprising. What is interesting is that the number of simple arcs in a geodesic $\gamma$ is also bounded by $\sqrt{i(\gamma, \gamma)}$.

4. Our theorem then has the form
   $$\#G^c_*(L, K) \leq c_* L^{c_* N(L,K)}$$
   where $c_*$ is a constant depending on $X_0$.

6. Seeing self-intersections of $\gamma$

   The flat structure $X_0$ on $S$ gives us a useful decomposition of $\gamma$ into saddle connections. However, geodesics in $X_0$ are generically not self-transverse. So the number of self-intersections of $\gamma$ is not well-defined. We approximate each $\gamma \in G^c_*(L, K)$ with a family of nearby curves $\{ \gamma_t \}$ so that
   $$\#\gamma_t \cap \gamma_t = i(\gamma, \gamma), \forall t$$

   In fact, we want to choose $\gamma_t$ to be a geodesic in some negatively curved metric $X_t$, for each $t$. For this, we need the following proposition.

**Proposition 6.1.** Given a flat metric $X_0$ on $S$ with one singular point $s$, there is a sequence of negatively curved metrics $\{ X_t \}$ so that

$$\lim_{t \to 0} X_t = X_0.$$
Proof. We start by approximating $X_0$ by a sequence of negatively curved metrics with a cone point at $s$. $X_0$ can be formed by gluing together the sides of some $4g$-gon, $A$. This is because we can cut $S$ along disjoint saddle connections until we get a flat polygon.

We want to approximate $A$ by $4g$-gons that have constant curvature $-t^2$ for each $0 < t < T$, for some $T$. For each $t > 0$, let $H_t^2$ be the hyperbolic plane, but with constant curvature $-t^2$. Cut $A$ into triangles $T_1, \ldots, T_{4g-2}$. For each $i$, take triangles $T_1^i, \ldots, T_{4g-2}^i$ in $H_t^2$ with the same side lengths as $T_1, \ldots, T_{4g-2}$. The side lengths uniquely determine the triangles up to isometry. Thus, $\lim_{t \to 0} T_i^t = T_i$, for each $i$. Glue the triangles in $H_t^2$ together to get a $4g$-gon $A_t$ in $H_t^2$ with the same side lengths as $A$. This ensures that $\lim_{t \to 0} A_t = A$.

We can glue together opposite sides of $A_t$ by isometries to get a metric $Y_t$ on $S$. Then $Y_t$ will have constant curvature $-t^2$ outside of the cone point $s$. This cone point has a cone angle that converges to the cone angle of $X_0$. Because $\lim A_t = A$, we get that $\lim_{t \to 0} Y_t = X_0$ on all compact sets outside of the singular point $s$.

Now for each $t$, we will cut out a disc $D_{\text{cut}}(t)$ about $s$, and glue in a smooth disc $D_{\text{glue}}(t)$. Let $D_{\text{cut}}(t)$ be a disc of radius $3t$. Take local polar coordinates $(r, \theta)$ on $D_{\text{cut}}$ so that $s$ lies at $r = 0$. We claim that in local coordinates, the metric looks like

$$Y_t = dr^2 + f_t(r)d\theta^2$$

where

$$f_t(r) = \frac{\alpha}{2\pi} \frac{1}{t} \sinh(tr)$$

and $\alpha$ is the cone angle at $s$. By, for example, [Pet06][Chapter 2, p.47] the curvature of a metric of this form is $-\frac{\alpha^2}{2\pi^2}$. So we see that the curvature of this metric is $-t^2$. This metric is singular only at $s$. To compute the angle at $s$, we will compute instead the circumference, $c_t$, of a disc of radius $\epsilon$ about $s$:

$$c_t = \int_{\theta=1}^{2\pi} \frac{\alpha}{2\pi} \frac{1}{t} \sinh(tc)d\theta = \frac{\alpha}{t} \sinh(tc)$$

This is exactly the circumference of a wedge with angle $\alpha$ and radius $\epsilon$. (For example, the circumference of a circle of radius $\epsilon$ in $H_t^2$ is $2\pi \frac{1}{t} \sinh(tc)$.) Therefore, this is the correct metric.

Now we want to take a disc $D_{\text{glue}}$ with metric $dr^2 + g_t(r)d\theta$ so that for some $0 < t_0 < 2t$,

- $g_t(r) = \begin{cases} \frac{1}{2\pi} \sinh tr & \text{if } r \in [0, t_0] \\ \frac{1}{2\pi} \sinh tr & \text{if } r \in [2t, 3t] \end{cases}$

and

- $g_t(r)$ is smooth and convex on $[0, 3t]$.

To construct $g_t(r)$, we draw the tangent line $l_{\alpha}(r)$ to $f_{\alpha}(r) = \frac{\alpha}{2\pi} \frac{1}{t} \sinh(tr)$ at $t$. Since $f_{\alpha}(0) = 0$ and $f_{\alpha}$ is convex, there is some $0 < s_0 < t$ so that $l_{\alpha}(s_0) = 0$. Draw the tangent line $l_1(r)$ to $f_1(r)$ at $s_0$. Since $f_1$ is convex, and $f_{\alpha} = \frac{\alpha}{2\pi} f_1$ for $\alpha > 2\pi$, there is some time $s_0 < s_1 < t$ so that $l_{\alpha}(s_1) = l_{\alpha}(s_1)$. Consider

$$h_t(r) = \begin{cases} \frac{1}{t} \sinh tr & \text{if } r \in [0, s_0] \\ l_1(r) & \text{if } r \in [s_0, s_1] \\ l_{\alpha}(r) & \text{if } r \in [s_1, t] \\ \frac{\alpha}{2\pi} \frac{1}{t} \sinh tr & \text{if } r \in [t, 3t] \end{cases}$$
Then $h_t(r)$ is convex, but not smooth at $s_0, s_1$ or $t$. However, by [Gho02], given any $\delta > 0$, there is some function $g_t(r)$ that is smooth and equal to $h_t(r)$ outside of $\delta$ neighborhoods of $s_0, s_1$ and $t$. In particular, we can find a function $g_t(r)$ and a radius $t_0 < s_0$ so that $g_t(r)$ satisfies the conditions above.

Because $g_t(r)$ is convex, $D_{\text{glue}}$ has negative curvature everywhere. And because of the way that we defined $g_t(r)$, the metric near the boundary of $D_{\text{glue}}$ matches up with the metric near the boundary of $D_{\text{cut}}$. So we can glue it in to $(S, Y_t) \setminus D_{\text{cut}}$ to get a new negatively curved metric $X_t$ on $S$ that is smooth at $\partial D_{\text{glue}}$.

Next, we see that the angle of $X_t$ at $s$ is $2\pi$. This is because near $r = 0$, $g_t(r)$ is just like $f_t$ but with $\alpha$ replaced with $2\pi$. So locally near $s$, $D_{\text{glue}}$ looks like a smooth disk of constant curvature $-t^2$.

The last thing we need to check is that the area of $D_{\text{glue}}$ goes to zero as $t$ goes to infinity. This will ensure that $\lim_{t \to 0} X_t = X_0$ on all compact sets outside of $s$. The area of $D_{\text{glue}}$ is given by

$$\text{Area}_t = \int_0^{3t} \sqrt{g_t(r)} dr \wedge d\theta$$

We know that $g_t(r)$ is increasing on $[0, 3t]$ because it is a convex function that is increasing between 0 and $t_0$. So its maximum value is $\frac{\alpha}{2\pi t} \sinh(3t^2)$. As $\lim_{t \to 0} \frac{1}{t} \sinh(3t^2) = 0$ uniformly, there is some $\epsilon$ small enough so that for all $t < \epsilon$,

$$\frac{\alpha}{2\pi t} \sinh(3t^2) < 1$$

Thus, $\text{Area}_t < 6\pi t$ for all $t < \epsilon$. So, $\lim_{t \to 0} \text{Area}_t = 0$. Since $D_{\text{glue}}$ is a disc, its radius goes to zero if its area goes to zero. So

$$\lim_{t \to 0} X_t = X_0$$

on all compact sets outside of $s$, and $X_t$ is a smooth, negatively curved metric for each $t$.

This proposition allows us to approximate geodesics on $X_0$ by geodesics in nearby negatively curved metrics.

**Lemma 6.2.** For each $\gamma \in G_*$, there is a continuous family of curves $\{\gamma_t\}_{t \in [0, T]}$, with $\gamma_0 = \gamma$ and so that $\gamma_t$ is a geodesic in a negatively curved metric space $X_t$ for each $t$. In particular,

$$i(\gamma, \gamma) = \#\gamma_t \cap \gamma_t, \forall t \in [0, T]$$

**Proof.** Take a sequence of negatively curved metrics $X_t$, where $\lim_{t \to 0} X_t = X_0$. For each $t$, $\gamma$ is freely homotopic to a closed $X_t$-geodesic $\gamma_t$. So, $\lim_{t \to 0} \gamma_t = \gamma$, pointwise. Because $\gamma$ has finite length, $\{\gamma_t\}_{t \in [0, T]} \cup \{\gamma\}$ is a continuous family of curves. Geodesics in negatively curved metrics realize self-intersection number, so $i(\gamma, \gamma) = \#\gamma_t \cap \gamma_t, \forall t$. □

We wish to control how close to $\gamma$ these approximations are. For this we need the following definition.

**Definition 6.3.** Two closed curves (or two arcs) $\gamma$ and $\gamma'$ are $\epsilon$-homotopic if there is a homotopy between them that moves each point a distance of at most $\epsilon$. Then we write $\gamma \sim_\epsilon \gamma'$. 
Remark 6.4. Being $\epsilon$-homotopic is reflexive and symmetric, but not transitive. In fact, if $\gamma_1 \sim_\epsilon \gamma_2$ and $\gamma_2 \sim_\epsilon \gamma_3$ then $\gamma_1 \sim_\epsilon \gamma_3$.

The problem with approximating a flat geodesic $\gamma$ with an $X_t$-geodesic $\gamma_t$ that realizes its self-intersection number, is that $\gamma_t$ is no longer naturally decomposed into saddle connections. The following lemma gives a way to decompose $\gamma_t$ into approximations of saddle connections.

Lemma 6.5. Fix $L$. There is an $\epsilon_L$ depending only on $L$ so that the following holds for all $\epsilon \leq \epsilon' < \epsilon_L$. Let $D_{\epsilon'}$ be an $\epsilon'$-neighborhood of the singular point $s$. For any $\gamma \in \mathcal{G}_t^c(L)$, there is a curve $\gamma_\epsilon \sim_\epsilon \gamma$ for which $i(\gamma, \gamma) = \# \gamma \cap \gamma_\epsilon$ and for which we can write

$$\gamma_\epsilon = s_1 \circ d_1 \circ \ldots s_n \circ d_n$$

where $s_i \subset S \setminus D_{\epsilon'}$ and $d_i \subset D_{\epsilon'}$.

Furthermore, suppose $\gamma = \sigma_1 \ldots \sigma_n$, with $\sigma_i \in \mathcal{C}, \forall i$. Then for each $i$,

$$s_i \sim_2 \sigma_j \iff \sigma_j = \sigma_i.$$

(See Figure 1)

![Figure 1](image-url)  

**Figure 1.** How to approximate $\gamma$, while retaining information about saddle connections. The arcs $d_1$, $d_2$ and $d_3$ lie in the shaded disc $D_{\epsilon'}$.

Proof. First we choose $\epsilon_L$. For each $\epsilon$, let $D_{\epsilon}$ be the disc of radius $\epsilon$ about $s$. Because there are finitely many saddle connections of length at most $L$, there is some distance $\epsilon'_L$ so that if $\epsilon < \epsilon_L$ and if $\sigma \in \mathcal{C}$ with $l_0(\sigma) \leq L$, then $\sigma$ crosses $\partial D_{\epsilon}$ exactly twice. Let $l_0$ be the length of the shortest closed geodesic on $X_0$. Then we set

$$\epsilon_L = \min\{\epsilon'_L, \frac{l_0}{8}\}$$

Choose $\epsilon$ and $\epsilon'$ so that $\epsilon \leq \epsilon' < \epsilon_L$, and take the disc $D_{\epsilon'}$.

Take a continuous family $\{\gamma_t\}$, for $t \in [0, T]$, where $\gamma_0 = \gamma$ and where, for each $t$, $\gamma_t$ is a geodesic in negatively curved metric $X_t$ on $S$ (from Lemma 6.2.) There is some $t_0$ depending on $\epsilon$ so that for all $t \leq t_0$, $\gamma_t \sim_\epsilon \gamma$, and the homotopy on $[0, t_0]$ is transverse to $\partial D_{\epsilon'}$. Thus, the number of intersections of $\gamma_t$ with $\partial D_{\epsilon'}$ remains constant for all $t \in [0, t_0]$. Define

$$\gamma_\epsilon = \gamma_{t_0}$$
Write $\gamma = \sigma_1 \ldots \sigma_n$, for $\sigma_i \in \mathcal{C}, \forall i$. For each $\sigma_i$, the homotopy $\{\gamma_t\}_{t \in [0, \epsilon_0]}$ gives a correspondence between $\sigma_i$ and a subarc $(\sigma_i)_\epsilon$ of $\gamma_\epsilon$. Since the homotopy moves each point of $\gamma$ by at most $\epsilon < \epsilon'$, the endpoints of $(\sigma_i)_\epsilon$ lie inside $D_\nu$. Because $\epsilon < \epsilon_L$, each saddle connection crosses $\partial D_\nu$ exactly twice. Because the homotopy $\{\gamma_t\}$ is transverse to $\partial D_\nu$, the arc $(\sigma_i)_\epsilon$ also crosses $\partial D_\nu$ exactly two times. Let $s_i$ denote the part of $(\sigma_i)_\epsilon$ outside of $D_\nu$. We get that

$$\gamma_\epsilon = s_1d_1 \ldots s_n d_n$$

where $s_1, \ldots, s_n$ are the arcs defined above, and $d_i$ connects $s_{i-1}$ to $s_i$. Because $\gamma_\epsilon$ crosses $\partial D_\nu$ only at the endpoints of $s_1, \ldots, s_n$, each $d_i$ must be contained inside $D_\nu$.

Now we need to show that $s_i \sim_{2\epsilon} \sigma_j$ if and only if $\sigma_j = \sigma_i$. Because $\sigma_i \sim_{\epsilon} (\sigma_i)_\epsilon$ and because $s_i \sim_{\epsilon} (\sigma_i)_\epsilon$, we have that $\sigma_i \sim_{2\epsilon} s_i$.

Suppose $\sigma_j \sim_{2\epsilon} s_i$ for some $j$. Then $\sigma_j \sim_{4\epsilon} \sigma_i$. If $\sigma_i \neq \sigma_j$, the $4\epsilon$ homotopy between them sends some endpoint of $\sigma_i$ to an endpoint of $\sigma_j$ along a non-trivial loop based at $s$. This loop can be tightened to a closed geodesic, whose length must be at least $l_0$. By assumption, $4\epsilon < \frac{l_0}{2}$. Thus, two saddle connections are $4\epsilon$-homotopic if and only if they are equal. Therefore, $\sigma_j \sim_{2\epsilon} s_i$ if and only if $\sigma_i = \sigma_j$.

7. Intersection number for arcs

Take a geodesic arc $\delta : s \mapsto s$. We want to define a geodesic self-intersection number $i(\delta, \delta)$ that is intrinsic to $\delta$. This intersection number should have the following property.

Suppose $\gamma$ is a geodesic in $X_0$, and take some curve $\gamma_\epsilon \sim_\epsilon \gamma$. Suppose our arc $\delta$ happens to be a subarc of $\gamma$. The homotopy from $\gamma$ to $\gamma_\epsilon$ gives a correspondence between $\delta$ and some subarc $\delta_\epsilon$ of $\gamma_\epsilon$. As long as $\epsilon$ is small enough, we want

$$i(\delta, \delta) \leq \# \delta_\epsilon \cap \delta_\epsilon$$

where the left hand side is the intrinsic self-intersection number defined below, and the right hand side is the number of intersections we observe in $\delta_\epsilon$. This is formalized in the following definition:

**Definition 7.1.** Let $\delta_1, \delta_2 : s \mapsto s$ be two geodesic arcs in $X_0$. For each $\epsilon > 0$, let

$$i_\epsilon(\delta_1, \delta_2) = \inf \{ \# (\delta_1)_\epsilon \cap (\delta_2)_\epsilon \mid \delta_i \sim_\epsilon (\delta_i)_\epsilon, i = 1, 2 \}$$

and let

$$i(\delta_1, \delta_2) = \lim_{\epsilon \to 0} i_\epsilon(\delta_1, \delta_2)$$

Note that when $\delta_1 = \delta_2$, we require $(\delta_1)_\epsilon = (\delta_2)_\epsilon$, and we count the number of transverse self-intersections of $(\delta_1)_\epsilon$. Thus, a **simple geodesic arc** is one whose self-intersection number is zero in this sense. (See Figure 2.)

**Remark 7.2.** The limit $\lim_{\epsilon \to 0} i_\epsilon(\delta_1, \delta_2)$ exists: If $\epsilon < \epsilon'$, and $\delta_\epsilon \sim_\epsilon \delta$ for some geodesic arc $\delta : s \mapsto s$, then $\delta_\epsilon \sim_{\epsilon'} \delta$, too. For this reason, $i_\epsilon(\delta_1, \delta_2)$ is a decreasing function of $\epsilon$. It is bounded below by zero. Thus the limit $\lim_{\epsilon \to 0} i_\epsilon(\delta_1, \delta_2)$ must exist.
8. Counting simple arcs

Let $\mathcal{C}_0$ be the set of simple geodesic arcs:

$$\mathcal{C}_0 = \{ \delta : s \mapsto s \text{ geodesic} \mid i(\delta, \delta) = 0 \text{ as an arc} \}$$

and consider those simple arcs of length less than $L$:

$$\mathcal{C}_0(L) = \{ \delta \in \mathcal{C}_0 \mid l(\delta) \leq L \}$$

**Lemma 8.1.** Fix an $L$. Then we get the following upper bound on the size of $\mathcal{C}_0(L)$:

$$\# \mathcal{C}_0(L) \leq (c_0 L)^{c_g}$$

where $c_0$ is a constant depending only on the geometry of $X_0$ and $c_g$ is a constant depending only on the surface $S$.

**Proof.** We will fix an $\epsilon'$ for the proof of this lemma (and for all the claims used to prove it). There is some $\mu_L > 0$ depending only on $L$ so that $\forall \epsilon' < \mu_L, \forall \sigma_1, \sigma_2 \in \mathcal{C}$ with $l(\sigma_i) < L, i = 1, 2$, we get that $\sigma_1$ and $\sigma_2$ do not intersect on $\partial D_{\epsilon'}$. Such a $\mu_L$ exists because the set $\{ \sigma \in \mathcal{C} \mid l(\sigma) < L \}$ is finite, so the set of intersection points between pairs $\sigma_1, \sigma_2$ of saddle connections in this set is also finite. In other words, there is some disk about $s$ of radius $2\mu_L$ that contains no intersection points between pairs of saddle connections in this set. Let $\epsilon_L$ be the constant from Lemma 6.5. Now choose any

$$\epsilon' < \min\{\epsilon_L, \mu_L\}$$

The proof of Lemma 8.1 goes as follows.

- We fix some set $\Sigma = \{\sigma'_1, \ldots, \sigma'_m\}$ of distinct saddle connections, and consider those $\delta \in \mathcal{C}_0(L)$ composed only of saddle connections in $\Sigma$. That is, we set

$$\mathcal{C}_0(L, \Sigma) = \{ \delta = \sigma_1 \ldots \sigma_n \in \mathcal{C}_0(L) \mid \forall i, \sigma_i \in \Sigma \}$$

Then we bound the size of this set (Claim 8.2.) We use techniques that are similar to those found in [BS85]. Roughly, a geodesic in $\mathcal{C}_0(L, \Sigma)$ is given by weights on the arcs in $\Sigma$, together with data that give the order in which the arcs are traversed. The bound on length gives restrictions on which weights are possible. The fact that arcs in $\mathcal{C}_0(L, \Sigma)$ are simple restricts the order in which the saddle connections can be traversed. This allows us to bound the size of $\mathcal{C}_0(L, \Sigma)$. 

![Figure 2](image-url) As an arc, $\delta$ is simple, even though it has one self-intersection as a closed curve.
Claim 8.2. Let \( \Sigma = \{ \sigma'_1, \ldots, \sigma'_m \} \) be a set of distinct saddle connections with \( l_0(\sigma'_i) \leq L, \forall i \).

Let \( C_0(L, \Sigma) = \{ \delta = \sigma_1 \ldots \sigma_n \in C_0(L) \mid \forall i, \sigma_i \in \Sigma \} \) be the set of \( \delta \in C_0(L) \) composed of the saddle connections in \( \Sigma \). Then

\[
\#C_0(L, \Sigma) \leq 16 \left( \frac{L}{l_0} \right)^{m^2+4}
\]

where \( l_0 \) is the length of the shortest closed geodesic on \( X_0 \).

Proof: This argument is inspired by techniques from the proof of a theorem of Birman and Series [BS85].

Consider the points \( x_1, \ldots, x_{2m} \) where the saddle connections in \( \Sigma \) intersect \( \partial D_{e'} \).

We have fixed an \( e' < \min\{ \epsilon_L, \mu_L \} \) at the start of the proof of Lemma 8.1. Because \( l_0(\sigma'_i) \leq L \) for each \( i \), our choice of \( e' \) guarantees that \( x_1, \ldots, x_{2m} \) are all distinct. Let \( I_r(x_i) \) be the ball of radius \( r \) about \( x_i \) in \( \partial D_{e'} \). Choose \( r \) small enough so that \( I_r(x_i) \) and \( I_r(x_j) \) are disjoint for each \( i \neq j \). From now on, let 

\[
I_i = I_r(x_i)
\]

Suppose \( \delta \in C_0(L, \Sigma) \). Write \( \delta = \sigma_1 \ldots \sigma_n \), for \( \sigma_i \in \mathcal{C}, \forall i \). Let \( \epsilon = \min\{ \frac{\epsilon}{2}, e' \} \).

The proof of Lemma 6.5 never used the fact that \( \gamma \) was closed. By assumption, \( \epsilon \leq e' < \epsilon_L \), where \( \epsilon_L \) is the number from Lemma 6.5. So the lemma implies that there is an arc \( \delta \sim \delta \) so that \( \#\delta \cap \delta = 0 \) and that we can write as

\[
\delta = s_1d_1 \ldots d_{n-1}s_n
\]

where \( s_i \) lies outside the disc \( D_{e'} \), \( s_i \sim_{2e} \sigma_i \), and \( d_i \) lies inside \( D_{e'} \), for each \( i \). (See Figure 3, but ignore the caption for now.)

Suppose \( \sigma_i \) has endpoints \( x_{j_i} \) and \( x_{k_i} \). Since \( s_i \sim_{2e} \sigma_i \), and \( \epsilon < \frac{\epsilon}{2} \), the endpoints of \( s_i \) lie in \( I_{j_i} \) and \( I_{k_i} \). Thus, the arcs \( s_1, \ldots, s_n \) connect the intervals \( I_1, \ldots, I_m \) outside \( D_{e'} \), and the arcs \( d_1, \ldots, d_{n-1} \) connect these intervals inside \( D_{e'} \) (Figure 3).

Now we are ready to give combinatorial data that encodes how many times each \( \sigma'_i \) appears in \( \delta \), as well as the order of the saddle connections inside \( \delta \). Let \( n_{i,j} \) be the number of arcs that connect \( I_i \) to \( I_j \) inside \( D_{e'} \). Number the intersection points of \( \delta \) with \( \partial D_{e'} \) clockwise from some fixed endpoint of \( I_1 \). Let \( t_0 \) and \( t_1 \) be
the number of the start- and endpoints of \( \delta \), respectively. Let \( I_{i_0} \) and \( I_{i_1} \) be the intervals that contain the start- and endpoints of \( \delta \), respectively. Then \( \delta \) has data \( D(\delta) = \{ \{ n_{ij} \}, t_0, t_1, i_0, i_1 \} \). (See Figure 3.)

We will show that only \( \delta \) can have data \( D(\delta) \). First of all, the data determines the number of times each saddle connection \( \sigma^j_i \in \Sigma \) appears in \( \delta \). For each \( i \), let

\[
n_i = \sum_j n_{ij}
\]

Suppose \( \sigma^j_i \) has an endpoint on \( I_{j_k} \). If \( j_k \neq i_0, i_1 \), then \( \sigma^j_i \) appears \( n_{j_k} \) times. Otherwise, it appears either \( n_{j_k} + 1 \) or \( n_{j_k} + 2 \) times, depending on whether just one of \( i_0 \) and \( i_1 \) is \( j_k \), or if \( i_0 = i_1 = j_k \), respectively.

If \( \delta' \) has the same data as \( \delta \) then we have shown that the saddle connections in \( \Sigma \) appear in \( \delta \) and \( \delta' \) with the same multiplicity. We need to show that the saddle connections also appear in the same order. This will imply \( \delta = \delta' \).

Take an arc \( \delta'_{\epsilon'} \sim_{\epsilon} \delta' \), that we can write \( \delta'_{\epsilon'} = s'_{1}d'_{1} \ldots d'_{n-1}s'_{n} \), where \( s'_{i} \) lies outside \( D_{\epsilon'} \) and \( d'_{i} \) lies inside \( D_{\epsilon'} \). Suppose \( \delta'_{\epsilon'} \) intersects \( \partial D_{\epsilon'} \) at points \( y_0, \ldots, y_{2n} \). Suppose the indices on these points correspond to their order around \( D_{\epsilon'} \). We will show that we can recover the order in which \( y_0, \ldots, y_{2n} \) appear in \( \delta'_{\epsilon'} \) just from the data.

The arc \( \delta'_{\epsilon'} \) gives a pairing of the set of points \( \{ y_0, \ldots, y_{2n} \} \) by arcs inside \( D_{\epsilon'} \) and a pairing of the set of points \( \{ y_0, \ldots, y_{2n} \} \) by arcs outside \( D_{\epsilon'} \). We will actually show that we recover both of these pairings. This will give us the order in which \( y_0, \ldots, y_{2n} \) appear in \( \delta'_{\epsilon'} \).

Since \( \delta' \) has the same data as \( \delta \), it also has \( n_{ij} \) of the arcs in \( \{ d'_1, \ldots, d'_{n-1} \} \) connecting points on \( I_i \) to points on \( I_j \). Because \( D_{\epsilon'} \) is a disc, there is only one way to pair the points by disjoint arcs inside \( D_{\epsilon} \) so that \( n_{ij} \) points on \( I_i \) are joined to points on \( I_j \). Therefore, the data determines the pairing of points inside \( D_{\epsilon'} \).

Now we turn to the pairing of points by arcs outside \( D_{\epsilon'} \). Suppose a saddle connection \( \sigma^j_i \in \Sigma \) joins interval \( I_{j_k} \) to interval \( I_{k_i} \). Then the points in \( \{ y_0, \ldots, y_{2n} \} \) that lie on \( I_{j_k} \) can only be paired to those points that lie on \( I_{k_i} \). These points are paired by a set of disjoint arcs outside \( D_{\epsilon'} \) that are \( \epsilon \)-homotopic to \( \sigma^j_i \). We claim that only one pairing by disjoint arcs is possible. The proof of this is a bit technical, but roughly speaking all we do is lift everything to the universal cover to reduce this
to a problem of connecting points on the boundary of a simply connected domain.
(See Figure 4.)

Let \( N_{\epsilon}(\sigma_i) \) be an \( \epsilon \)-neighborhood of \( \sigma_i' \). If \( s_j' \sim_{2\epsilon} \sigma_i' \), then \( s_j' \in N_{\epsilon}(\sigma_i') \). (This follows from the construction of \( s_1', \ldots, s_n' \) in Lemma 6.5.) Lift \( N_{\epsilon}(\sigma_i') \) to a region \( \tilde{N}_{\epsilon}(\sigma_i') \) in the universal cover. This is an \( \epsilon \)-neighborhood of some lift \( \tilde{\sigma}_i' \) of \( \sigma_i' \). There are two lifts \((\tilde{D}_c)_1\) and \((\tilde{D}_c)_2\) at either end of \( \tilde{N}_{\epsilon}(\sigma_i') \). Because \( \epsilon < \epsilon_L \), \((\tilde{D}_c)_1\) and \((\tilde{D}_c)_2\) are disjoint. In fact, \( \tilde{N}_{\epsilon}(\sigma_i') \) is composed of \((\tilde{D}_c)_1\), \((\tilde{D}_c)_2\), and a simply connected region \( \tilde{R} \) between them (see Figure 4). As \( s_1', \ldots, s_n' \) lie outside of \( D_c \) but inside \( N_{\epsilon}(\sigma_i') \), their lifts \( \tilde{s}_1', \ldots, \tilde{s}_n' \) lie in \( \tilde{R} \) and have endpoints on the boundaries of \((\tilde{D}_c)_1\) and \((\tilde{D}_c)_2\). Because \( \tilde{R} \) is simply connected, is just one way to join the endpoints of \( \tilde{s}_1', \ldots, \tilde{s}_n' \) lying on \( \partial(\tilde{D}_c)_1 \) and \( \partial(\tilde{D}_c)_2 \).

\[ \text{Figure 4. The } \epsilon \text{-neighborhood of } \tilde{\sigma}_i' \text{ is shaded. There is only one way to join the points on } \partial(\tilde{D}_c)_1 \text{ to the points on } \partial(\tilde{D}_c)_2 \text{ inside the shaded region.} \]

Therefore, the pairings on the sets \( \{y_0, \ldots, y_{2n}\} \setminus \{y_0, y_1\} \) and \( \{y_0, \ldots, y_{2n}\} \) by arcs inside and outside of \( D_c \), respectively, is determined by the \( D(\delta) \).

The order in which \( y_0, \ldots, y_{2n} \) appear in \( \delta \), tells us the order in which the saddle connections appear in \( \delta \). This is because \( D(\delta) \) tells us how many points lie on each interval, and the cyclic order of the points, and so it tells us which points lie on which intervals. Each interval determines a saddle connection, and so the order of the points determines the order of saddle connections. Since we recover order of the saddle connections from the data alone, there is just one simple arc with data \( D(\delta) \).

For example, consider Figure 3. If we number the intersection points of \( \delta \) with \( \partial D_c \) clockwise, starting at \( I_1 \), then the order of the intersection points on \( D_c \) is \( y_5, y_4, y_2, y_1, y_6, y_3 \). Suppose we have this information along with \( D(\delta) \) and we wish to recover \( \delta \). The first pair of points are always joined by an arc outside of \( D_c \), because \( \delta \) always starts with an arc outside \( D_c \) by construction. We deduce from \( D(\delta) \) that \( y_5 \in I_4 \). This tells us that \( y_5 \) and \( y_4 \) are joined by an arc homotopic to \( \sigma_1 \). Then \( y_4 \) and \( y_6 \) must be joined by an arc inside \( D_c \). The next pair of points is \( y_2 \) and \( y_1 \). We can deduce that \( y_2 \) lies on \( I_2 \). So the points \( y_2 \) and \( y_1 \) must be joined by an arc homotopic to \( \sigma_2 \). Continuing on like this, we recover that \( \delta = \sigma_1 \sigma_2 \sigma_1 \).

So we have shown that if two simple arcs \( \delta \) and \( \delta' \) give the same data, then \( \delta = \delta' \).

Each saddle connection on \( X_0 \) has length at least \( l_0 \). As \( l_0(\delta) \leq L \) for each \( \delta = \sigma_1 \ldots \sigma_n \in C_0(L, \Sigma) \), the number \( n \) of saddle connections must be at most \( \frac{L}{l_0} \). Given data \( D(\delta) = \{ n_{ij}, t_0, t_1, t_i, t_j \} \) for any \( \delta \in C_0(L, \Sigma) \), this implies that

\[ \sum_{i,j=1}^{m} n_{ij} \leq \frac{L}{l_0} \]
where \( m \) is the size of the set \( \Sigma \). This sum has at most \( m^2 \) terms. Thus number of sets \( \{ n_{ij} \} \) of this form is at most \( (\frac{L}{l_0})^{m^2} \). Furthermore, \( t_0, t_1, i_0 \) and \( i_1 \) are numbers between 1 and \( \frac{2L}{l_0} \), so there are at most \( (\frac{2L}{l_0})^4 \) choices for them. Thus, the number of possible sets of data given \( \sigma \) is at most
\[
2^4 \left( \frac{L}{l_0} \right)^{m^2 + 4}
\]
\[ \square \]

We now work on understanding the sets \( \Sigma \) of distinct saddle connections that occur together in a simple arc of length bounded by \( L \).

**Claim 8.3.** Let \( \Sigma = \{ \sigma'_1, \ldots, \sigma'_m \} \) be the set of distinct saddle connections that can occur in some \( \delta \in C_0(L) \). Then the size of the set is bounded by
\[
m \leq b_g
\]
where \( b_g \) depends only on the topology of \( S \).

**Proof.** This claim follows from the following claim.

**Claim 8.4.** Let \( \delta \in C_0 \). Suppose \( \delta = \sigma_1 \ldots \sigma_n, \sigma_i \in C, \forall i \). Close up \( \sigma_i \) to a simple closed geodesic \( \bar{\sigma}_i \), for each \( i \). Then \( i(\bar{\sigma}_i, \bar{\sigma}_j) \leq 1 \) for each \( i \) and \( j \).

**Proof.** Let \( \epsilon_L \) be the constant defined in Lemma 6.5. Choose \( \epsilon < \epsilon_L \) and take some arc \( \delta \sim \delta \) given by Lemma 6.5. Then it is of the form \( \delta = s_1d_1 \ldots s_n d_n \), where \( s_i \) is an arc lying outside the disk \( D_\epsilon \) and \( d_i \) is an arc inside \( D_\epsilon \), for each \( i \).

Consider two saddle connections \( \sigma_i \) and \( \sigma_j \) of \( \delta \). They are \( 2\epsilon \)-homotopic as arcs to the arcs \( s_i \) and \( s_j \), respectively. Because \( \delta \) is simple, the arcs \( s_i \) and \( s_j \) are disjoint. We can join the endpoints of \( s_i \) inside the disc \( D_\epsilon \) to get a closed curve \( \bar{s}_i \) that is \( \epsilon \)-homotopic to \( \bar{\sigma}_i \) as a simple closed curve. We can do the same thing to get a closed curve \( \bar{s}_j \) from \( s_j \). Then \( \bar{s}_i \) and \( \bar{s}_j \) can only intersect inside \( D_\epsilon \). As \( D_\epsilon \) is simply connected, we can always arrange it so that \( \# \bar{s}_i \cap \bar{s}_j \leq 1 \). As \( \bar{s}_i \) and \( \bar{s}_j \) are freely homotopic to \( \bar{\sigma}_i \) and \( \bar{\sigma}_j \), respectively, we get that \( i(\sigma_i, \sigma_j) \leq 1 \). (See Figure 5).

![Figure 5](image-url)

**Figure 5.** Two saddle connections can be disjoint as arcs, but intersect when closed up. Here, \( i(\sigma_1, \sigma_2) = 0 \), but \( i(\bar{\sigma}_1, \bar{\sigma}_2) = 1 \). 

\[ \square \]
Now suppose $\delta \in \mathcal{C}_0$. Suppose $\Sigma = \{\sigma_1', \ldots, \sigma_m'\}$ is the set of distinct saddle connections occurring in $\delta$. We have $i(\sigma_i', \sigma_j') \leq 1$ for each $i, j$. Any set of distinct simple closed curves on $S$ with pairwise intersection at most one has size $O(g^3)$, where $g$ is the genus of $S$, by [Prz14]. In other words, there is a constant $b_g$ depending only on $S$ so that $\# \Sigma \leq b_g$.

**Claim 8.5.** The number of sets $\Sigma = \{\sigma_1', \ldots, \sigma_m'\}$ that can occur as sets of distinct saddle connections in arcs $\delta \in \mathcal{C}_0$ is at most

$$(b_0 L)^{2b_g}$$

where $b_g$ is a constant depending only on $S$, and $b_0$ is a constant depending on $X_0$.

**Proof.** If a saddle connection $\sigma$ occurs in some $\delta \in \mathcal{C}_0$, then $l_0(\sigma) \leq L$. The number of saddle connections on $X_0$ of length at most $L$ is bounded above by $b_0 L^2$, where $b_0$ is a constant depending only on $X_0$ [Mas90]. For any set $\Sigma$ of distinct saddle connections occurring in some $\delta \in \mathcal{C}_0$, $\# \Sigma \leq b_g$. The number of ways to choose $b_g$ elements from a set of size at most $b_0 L^2$ is at most $(b_0 L^2)^{b_g}$, which is bounded above by $(b_0 L^2)^{b_g}$.

Combining Claims 8.2 and 8.3, we get that

$$\# \mathcal{C}_0(L, \Sigma) \leq \left(\frac{L}{l_0}\right)^{d_g}$$

for each set $\Sigma$ of distinct saddle connections that occur in some $\delta \in \mathcal{C}_0(L)$, and where $d_g = b_g^2 + 4$ is a constant depending only on $S$. By Claim 8.5, there are at most $(b_0 L)^{2b_g}$ choices for $\Sigma$ so summing $\# \mathcal{C}_0(L, \Sigma)$ over all sets $\Sigma$ we get that

$$\# \mathcal{C}_0(L) \leq c_0 L^{c_g}$$

where $c_0 = \frac{b_0^{2b_g}}{l_0^{d_g}}$ and $c_g = d_g + 2b_g$.

**9. Bounding the number of simple arcs in a closed geodesic**

Suppose $\gamma = \sigma_1 \ldots \sigma_n \in \mathcal{G}^c_\infty$, for $\sigma_i \in \mathcal{C}$ for each $i$. Then we can take the smallest partition of $\sigma_1, \ldots, \sigma_n$ into simple arcs:

$$\gamma = \delta_1 \ldots \delta_m$$

for $\delta_1 = \sigma_1 \ldots \sigma_{n_1}$, $\delta_2 = \sigma_{n_1+1} \ldots \sigma_{n_2}$, and so on, with $\delta_i \in \mathcal{C}_0$ for each $i$. In particular, the arc $\delta_i \delta_{i+1}$, where we concatenate $\delta_i$ and $\delta_{i+1}$, is not simple.

If $\gamma \in \mathcal{G}^c_\infty(L)$, then each of the arcs $\delta_1, \ldots, \delta_m$ has length at most $L$. We know how to count simple arcs of length at most $L$, so now we bound the length $m$ of the sequence of these arcs in terms of $l_0(\gamma)$ and $i(\gamma, \gamma)$.

**Lemma 9.1.** Let $\gamma \in \mathcal{G}^c_\infty(L, K)$, with $K \geq 1$. Suppose

$$\gamma = \delta_1 \ldots \delta_m$$

is the shortest way to write $\gamma$ as a concatenation of arcs $\delta_i \in \mathcal{C}_0$. Then

$$m \leq \min\left\{\frac{L}{l_0}, cvK\right\}$$
where $l_0$ is the length of the shortest closed geodesic on $X_0$ and $c$ is a constant depending only on the topology of $S$.

Proof. Bounding $m$ in terms of $l_0(\gamma)$ is relatively simple. The difficulty lies in bounding $m$ in terms of $i(\gamma, \gamma)$, which we do first.

Suppose $\gamma \in \mathcal{G}_c(L, K)$ with $\gamma = \delta_1 \ldots \delta_m$ and $\delta_i \in \mathcal{C}_0$, $\forall i$. Suppose this is the shortest way to represent $\gamma$ as a concatenation of simple arcs. Then, as previously mentioned, the arc $e_i = \delta_i \delta_{i+1}$ is not simple. In fact, let

$$C_1 = \{ e = dd' | d, d' \in \mathcal{C}_0, i(e, e) \geq 1, e \text{ a geodesic arc} \}$$

be the set of non-simple concatenations of simple arcs. Moreover, let

$$C_2 = \{ f = ee' | e, e' \in C_1, f \text{ a geodesic arc} \}$$

Thus, each arc $f = ee' \in C_2$ has at least two self-intersections, one from $e$ and one from $e'$. Let $C_1(L)$ and $C_2(L)$ be the arcs in $C_1$ and $C_2$, respectively, that have length at most $L$. For our $\gamma$, let $f_i = e_i e_{i+2}$. The arcs $f_1, \ldots, f_m$ are well-defined as long as $m \geq 4$. If $m < 4$, then the lemma holds for any constant $c \geq 3$, because $K \geq 1$.

It turns out to be easier to bound $m$ using the non-simple arcs $f_1, \ldots, f_m$ rather than the simple arcs $\delta_1, \ldots, \delta_m$. We will show the following:

- Let $F = \{ f_1, \ldots, f_m \}$. We exhaust $F$ by sets $\mathcal{F}_1, \ldots, \mathcal{F}_N$, where each $\mathcal{F}_i$ is a maximal subset of pairwise disjoint arcs. We bound the size of $\mathcal{F}$ by bounding the size of each $\mathcal{F}_i$, and then by bounding their number, $N$.

- We show that

$$\#F_i \leq 2g - 2$$

for each $i$ (Lemma 9.2.) We do this by assigning each $f \in \mathcal{F}_i$ to either a pair of pants or a one-holed torus. Then we show that the set of pairs of pants and one-holed tori assigned to $\mathcal{F}_i$ are all distinct and are part of a pants decomposition of $S$. This is where we use that each $f \in C_2$ has at least two self-intersections.

- Now we want to show that the number $N$ of the sets $\mathcal{F}_1, \ldots, \mathcal{F}_N$ satisfies $N \leq c' \sqrt{K}$ for some universal constant $c'$. (This is proven as part of Lemma 9.7.) We have that

$$\sum_{i,j=1}^{m} i(f_i, f_j) \lesssim i(\gamma, \gamma)$$

where $A \lesssim B$ if $A \leq cB$ for some universal constant $c$. If each term $i(f_i, f_j)$ contributed at least 1 to this sum, we would be done. Unfortunately, this is not the case, precisely because we can find maximal disjoint subsets $\mathcal{F}_1, \ldots, \mathcal{F}_N$ of $F$. Fortunately, we can also say

$$\sum i(\mathcal{F}_i, \mathcal{F}_j) \lesssim i(\gamma, \gamma)$$

This is good because $i(\mathcal{F}_i, \mathcal{F}_j) \geq 1$ for each $i, j$, by the maximality of these sets. Therefore,

$$\sum_{i,j=1}^{N} 1 \lesssim i(\gamma, \gamma)$$

which implies that

$$N \leq c' \sqrt{K}$$
for some universal constant $c'$. 

- Combining the above two statements allows us to show that $\#F \leq cv\sqrt{K}$ for constant $c$ depending only on $S$ (Lemma 9.7).
- Lastly, we give a quick proof that if $\gamma = \delta_1, \ldots, \delta_n$, then $n \leq \frac{L}{l_0}$, where $l_0$ is the length of the shortest closed geodesic of $X_0$ to complete the proof (Section 9.3).

9.1. Maximal sets of pairwise disjoint, non-simple arcs. The following lemma tells us that a set of pairwise disjoint curves from $C_2$ cannot have very many elements.

**Lemma 9.2.** Fix a set $\{f_1, \ldots, f_m\}$ of arcs in $C_2$. Suppose $i(f_i, f_j) = 0, \forall i \neq j$. Then $m \leq 2g - 2$.

**Proof.** Because each $f_i$ has at least 2 self-intersections, there is a sense in which it fills either a pair of pants or one-holed torus $P_i$ (Claims 9.3 and 9.4). We then show that if $P_i$ and $P_j$ have an essential overlap as subsurfaces of $S$, then $f_i$ and $f_j$ must intersect (Claim 9.6). From this we deduce that the set $P_1, \ldots, P_m$ associated to $f_1, \ldots, f_m$ must be part of a pants decomposition of $S$, and therefore $m \leq 2g - 2$.

We start by choosing an $\epsilon$ and arcs $(f_i)_\epsilon \sim f_i$ that we will use for all the claims used to prove this lemma. Fix an $L$ so that $l_0(f_i) \leq L$ for each $i$. Because $\#|C_0(L) \cup C_1(L) \cup C_2(L)| < \infty$, there is some $\eta_L > 0$ so that $\forall \epsilon < \eta_L$, $\forall \delta \in C_0(L) \cup C_1(L) \cup C_2(L)$ and $\forall \delta \sim \epsilon \delta$, we have that $i(\delta, \delta) \leq \#\delta \cap \delta e$.

So choose $\epsilon < \min\{\epsilon_L, \eta_L\}$

where $\epsilon_L$ is the constant from Lemma 6.5.

For each $i$, choose arcs $(f_i)_\epsilon \sim f_i$ that are geodesics in a negatively curved metric, and so that for each $i$ and $j$, $i(f_i, f_j) = \#(f_i)_\epsilon \cap (f_j)_\epsilon$. Note that $i(f_i, f_j) = 0, \forall i \neq j$ implies that $(f_1)_\epsilon, \ldots, (f_m)_\epsilon$ are pairwise disjoint. Suppose each $(f_i)_\epsilon$ is parameterized as an arc, $(f_i)_\epsilon : [0, 1] \to S$. We find the $P_i$ using the following, rather technical, claim.

**Claim 9.3.** For each $i$, there is a closed sub-interval $I_i \subset [0, 1]$ so that

- $(f_i)_\epsilon|_{I_i}$ is a simple arc
- $(f_i)_\epsilon(\partial I_i) \subset (f_i)_\epsilon(I_i)$

In other words, each $(f_i)_\epsilon$ has a subarc with exactly two self-intersections, which looks like a figure eight. (See Figure 7.)

**Proof.** For each $f_i$, there are some arcs $e_i, e'_i \in C_1$ so that $f_i = e_i e'_i$. Thus, we can find subarcs $(e_i)_\epsilon$ and $(e_{i+1})_\epsilon$ of $(f_i)_\epsilon$ with disjoint domains so that $(e_i)_\epsilon \sim \epsilon e_i$, $(e'_i)_\epsilon \sim \epsilon e'_i$ of $(f_i)_\epsilon$, and

- $\#(e_i)_\epsilon \cap (e_i)_\epsilon \geq i(e_i, e_i) \geq 1$
- $\#(e'_i)_\epsilon \cap (e'_i)_\epsilon \geq i(e'_i, e'_i) \geq 1$

Because $(e_i)_\epsilon$ has at least one self-intersection point, it has a subarc $\alpha_i$ that can be closed up into a simple closed curve (See Figure 6.) That is, there are some $t_i < s_i \in [0, 1]$ so that

- $(f_i)_\epsilon|_{(t_i, s_i)}$ is simple
Figure 6. \(i(e_i, e_i) > 0\), so we can find a simple subloop \(\alpha_i \subset (e_i)_e\).

- \((f_i)_e(t_i) = (f_i)_e(s_i)\)

Let

\[ r_i = \min \{ r > s_i \mid \exists q, t_i < q < r, (f_i)_e(r) = (f_i)_e(q) \} \]

In other words, \(r_i\) is the first time after \(s_i\) that the arc starting at \(t_i\) loops back on itself.

Note that \(r_i\) exists. If it did not, then \((f_i)_e\) would be simple on the interval \((t_i, 1]\). But we know that \((e_i')_e\) is non-simple and its domain comes after the domain of \((e_i)_e\).

Let \(I_i = [t_i, r_i]\). Then \((f_i)_e(t_i) = (f_i)_e(s_i)\) and \((f_i)_e(r_i) = (f_i)_e(q_i)\) for some \(s_i, q_i \in (t_i, r_i)\). Furthermore, since we chose \(r_i\) to be minimal, \((f_i)_e\) must be simple on \((t_i, r_i)\). (See Figure 7.)

\[ \pi_i(r_i) = \pi_i(q_i) \]

\[ \pi_i(t_i) = \pi_i(s_i) \]

Figure 7. The figure eight subarc of \(f_i\), denoted \(\pi_i\).

\[ \square \]

For each \(i\), let

\[ \pi_i = (f_i)_e|_{I_i} \]

for the interval \(I_i\) from Claim 9.3. Let \(N(\pi_i)\) be a regular neighborhood of the graph of \(\pi_i\) in \(S\).

Claim 9.4. Either \(N(\pi_i)\) is a pair of pants or one-holed torus, and \(\partial N(\pi_i)\) is a set of essential curves.

NB: The curve \(\pi_i\) in Figure 7 fills a one-holed torus, while the curve \(\pi_i\) in Figure 8 fills a pair of pants.
Proof. By looking at the Euler characteristic of the graph of $\pi_i$, we see that $N(\pi_i)$ is either a pair of pants or a one-holed torus. To simplify notation, let $N(\pi_i) = \mathcal{P}_i$. We just need to show that no component of $\partial \mathcal{P}_i$ is null-homotopic.

There are two cases. First suppose that $\mathcal{P}_i$ is a one-holed torus with boundary curve $a$. If $a$ is null-homotopic then it bounds a disc $D$. This disc cannot lie in $\mathcal{P}_i$. So if we sew $D$ onto $\mathcal{P}_i$ at $a$, we will get a closed torus. But $\mathcal{S}$ is connected, and it is not a closed torus, so this is a contradiction.

Now suppose that $\mathcal{P}_i$ is a pair of pants. For what follows, refer to Figure 8.

![Figure 8. The case when $\mathcal{P}_i$ is a pair of pants](image)

Recall that we chose $(f_i)_\epsilon$ to be a geodesic arc in some negatively curved metric $X_\epsilon$. Take any point $x$ on the graph of $\pi_i$, and take a small disk $D$ around it. If we cut $D$ along $\pi_i$, one of two things can happen. If we chose one of the two endpoints of $\pi_i$, then $x$ is a point of intersection between an end of $\pi_i$ and a two-sided subarc of $\pi_i$. In this case, $D \setminus \pi_i$ has three components. The point $x$ lies on the boundaries of these components. In one of the components, the angle at $x$ is $180^\circ$ with respect to $X_\epsilon$, and in the other two, the angle is strictly smaller than $180^\circ$. If we choose any other point $x$, then $D \setminus \pi_i$ has two components that have $x$ on their boundaries, and the angle at $x$ is exactly $180^\circ$ on both of them.

Take $\mathcal{P}_i \setminus \pi_i$. Take the closure of each component of this set separately, and consider the disjoint union of these components. Abusing notation, we will still call the result $\mathcal{P}_i \setminus \pi_i$. Then $\mathcal{P}_i \setminus \pi_i$ has three cylindrical components $C_1, C_2$ and $C_3$. They each have boundary components, denoted $b_1, b_2$ and $b_3$, respectively, that lie on the graph of $\pi_i$. There are exactly four points $x_1, \ldots, x_4$ on $b_1 \cup b_2 \cup b_3$ where the angle inside $\mathcal{P}_i \setminus \pi_i$ at $x_j$ is smaller than $180^\circ$.

Each of $b_1, b_2$ and $b_3$ must contain one of $x_1, \ldots, x_4$. Otherwise, they would be simple closed geodesics. But this is impossible since the graph of $\pi_i$ does not contain a simple closed geodesic, as $\pi_i$ is not a simple closed geodesic itself. Thus, without loss of generality, $b_1$ contains $x_1$, $b_2$ contains $x_2$ and $b_3$ contains $x_3$ and $x_4$. Therefore, for each $j$, the sum of exterior angles around $b_j$ is strictly smaller than $360^\circ$. But if $b_j$ were null-homotopic, it would bound a disc. By Gauss-Bonnet, the
sum of exterior angles about the boundary of a disc in the negatively curved metric $X_\epsilon$ is greater than 360°. Thus, $b_1$, $b_2$ and $b_3$ are not null-homotopic.

Therefore, $\mathcal{P}_i$ is either an essential one-holed torus or an essential pair of pants embedded in $\mathcal{S}$.

□

From now on, we denote the neighborhood $N(\pi_i)$ by $\mathcal{P}_i$, for each $i$. So we have assigned each $f_i$ an essential pair of pants or one-holed torus $\mathcal{P}_i$. We want to show that $\mathcal{P}_1, \ldots, \mathcal{P}_m$ are distinct and that they are part of a pants decomposition of $\mathcal{S}$, which will imply that $m \leq 2g - 2$.

**Definition 9.5.** Given two sub-surfaces $S_1, S_2 \subset S$, we say $i(S_1, S_2) \neq 0$ if for any subsurfaces $S'_1$ and $S'_2$ isotopic to $S_1$ and $S_2$, respectively, $S'_1 \cap S'_2 \neq \emptyset$. Otherwise, we say $i(S_1, S_2) = 0$.

The following claim tells us that if $(f_1)_\epsilon, \ldots, (f_m)_\epsilon$ are all pairwise disjoint, then $\mathcal{P}_1, \ldots, \mathcal{P}_m$ can all be realized disjointly.

**Claim 9.6.** If $i(\mathcal{P}_i, \mathcal{P}_j) \neq 0$, then $\#\pi_i \cap \pi_j \geq 1$.

**Proof.** Suppose $i(\mathcal{P}_i, \mathcal{P}_j) \neq 0$. We will use the following fact to find intersections between $\pi_i$ and $\pi_j$: Because $\mathcal{P}_i = N(\pi_i)$ is a regular neighborhood of $\pi_i$, there is a deformation retract of $\mathcal{P}_i$ onto $\pi_i$. Thus, any closed curve in $\mathcal{P}_i$ is freely homotopic to a closed curve whose image lies in the graph of $\pi_i$. Consider $\partial \mathcal{P}_i$ and $\partial \mathcal{P}_j$ as multicurves. There are two cases to consider.

The first case is when $i(\partial \mathcal{P}_i, \partial \mathcal{P}_j) \neq 0$ (See Figure 9.) In this case, there are some boundary components $a \subset \partial \mathcal{P}_i$ and $b \subset \partial \mathcal{P}_j$ with $i(a, b) \neq 0$. Take closed curves $a'$ and $b'$ whose images in $\mathcal{S}$ lie inside the graphs of $\pi_i$ and $\pi_j$, so that $a'$ is freely homotopic to $a$ and $b'$ is freely homotopic to $b$. By definition, $i(a, b) \neq 0$ implies that $a' \cap b' \neq \emptyset$. But then, $\pi_i \cap \pi_j \neq \emptyset$ as well.

![Figure 9](image_url) The case when $\partial \mathcal{P}_i \cap \partial \mathcal{P}_j \neq \emptyset$.

Now consider the case when $i(\partial \mathcal{P}_i, \partial \mathcal{P}_j) = 0$ (See Figure 10.) Because $\mathcal{P}_i$ and $\mathcal{P}_j$ have an essential overlap as subsurfaces of $\mathcal{S}$, and because they are either pairs of pants or one-holed tori, we can say without loss of generality that $\mathcal{P}_i$ is isotopic to a subsurface of $\mathcal{P}_j$. (If $\mathcal{P}_i$ and $\mathcal{P}_j$ are both pairs of pants, or if they are both one-holed tori, then they would be isotopic. But if $\mathcal{P}_i$ is a pair of pants and $\mathcal{P}_j$ is a one-holed torus, then it is the closure of $\mathcal{P}_i$ inside $\mathcal{S}$ that is isotopic to $\mathcal{P}_j$.)
So we can choose a curve η with $i(\eta, \eta) = 1$ that can be isotoped to lie inside $\mathcal{P}_i$ and $\mathcal{P}_j$. Let $\eta_i$ and $\eta_j$ be the curves freely homotopic to $\eta$ that lie in the graphs of $\pi_i$ and $\pi_j$, respectively. As $i(\eta, \eta) = 1$, we have that $\# \eta_i \cap \eta_j \geq 1$. So $\# \pi_i \cap \pi_j \geq 1$. □

We have that $i(f_i, f_j) = 0, \forall i \neq j$. We chose $(f_1)_e, \ldots, (f_m)_e$ so that $\#(f_i)_e \cap (f_j)_e = i(f_i, f_j) = 0, \forall i \neq j$. For each $i$, $(f_i)_e$ corresponds to some pair of pants or one-holed torus $\mathcal{P}_i$ (Claim 9.4.) Choose any hyperbolic metric $X_{-1}$ on $S$, and let $N_1, \ldots, N_m$ be the pairs of pants and one-holed tori with geodesic boundary with respect to $X_{-1}$ so that $N_i$ is isotopic to $\mathcal{P}_i$ for each $i$. We have that $i(\mathcal{P}_i, \mathcal{P}_j) = 0$ if and only if $N_i \cap N_j = \emptyset$. Since $(f_i)_e$ and $(f_j)_e$ are pairwise disjoint for each $i \neq j$, $N_i$ and $N_j$ are disjoint for each $i \neq j$ (Claim 9.6.) Thus, $N_1, \ldots, N_m$ are part of some pants decomposition of $S$. Therefore, $m \leq 2g - 2$. □

9.2. Bound on number of simple arcs in terms of intersection number.
We are finally ready to bound the number of simple arcs in a geodesic $\gamma \in \mathcal{G}_c^*(L, K)$ in terms of its self-intersection number.

Lemma 9.7. Let $\gamma \in \mathcal{G}_c^*(L, K)$. Suppose

$$\gamma = \delta_1 \ldots \delta_m$$

is the shortest way to write $\gamma$ as a concatenation of arcs $\delta_1, \ldots, \delta_m \in \mathcal{C}_0$. If $K \geq 1$, then

$$m \leq c\sqrt{K}$$

where $c$ is a constant depending only on the topology of $S$.

Proof. Suppose $\gamma = \delta_1 \ldots \delta_m$, where $\delta_i \in \mathcal{C}_0, \forall i$. If $m \leq 3$, then we are done for $c \geq 3$. So suppose $m \geq 4$. Let $e_i = \delta_i \delta_{i+1}$. Then $e_i \in \mathcal{C}_1$. Now let $f_i = e_i e_{i+2}$. Then $f_i \in \mathcal{C}_2$. Both $e_i$ and $f_i$ are well-defined for each $i$ because there are at least four distinct simple arcs in $\gamma$.

First we show that

$$\sum_{i,j=1}^m i(f_i, f_j) \leq 32i(\gamma, \gamma)$$

Choose $\epsilon < \eta_L$, where $\eta_L$ is the constant defined at the start of the proof of Lemma 9.2. Take a curve $\gamma_\epsilon \sim \gamma$ so that $i(\gamma_\epsilon, \gamma) = \# \gamma_\epsilon \cap \gamma_\epsilon$.

The homotopy from $\gamma$ to $\gamma_\epsilon$ gives a correspondence between $f_i$ and some curve $(f_i)_\epsilon \subset \gamma_\epsilon$ for each $i$. If we parameterize $\gamma : [0, 1] \to S$ and let $J_1, \ldots, J_m$ be
the domains of $f_1, \ldots, f_m$, respectively, then each $t \in [0,1]$ lies in exactly 4 of $J_1, \ldots, J_m$. Thus, exactly 16 pairs $(f_i)_\epsilon, (f_j)_\epsilon$ intersect at each self-intersection point of $\gamma$. Because $\epsilon < \eta_L$, $\nu(f_i, f_j) \leq \#(f_i)_\epsilon \cap (f_j)_\epsilon$. Thus,

$$\frac{1}{2} \sum_{i, j} \nu(f_i, f_j) \leq 16 \nu(\gamma, \gamma)$$

where the $\frac{1}{2}$ comes from the fact that each pair $f_i, f_j$ appears twice in the sum on the left-hand side.

Let $\mathcal{F} = \{ f_1, \ldots, f_m \}$. We want to show $\nu(f_i, f_j) \neq 0$ for sufficiently many pairs $i, j$. To do this, we decompose $\mathcal{F}$ into small sets $\mathcal{F}_1, \ldots, \mathcal{F}_N$ for which $\nu(\mathcal{F}_i, \mathcal{F}_j) > 0$, $\forall i, j$. Up to renumbering indices, let

$$\mathcal{F}_1 = \{ f_1, \ldots, f_{n_1} \}$$

be a maximal subset of $\mathcal{F}$ so that $\nu(f_i, f_j) = 0$, $\forall i \neq j = 1, \ldots, n_1$. Given $\mathcal{F}_1, \ldots, \mathcal{F}_i$, let

$$\mathcal{F}_{i+1} = \{ f_{n_1+1}, \ldots, f_{n_{i+1}} \}$$

be a maximal subset of $\mathcal{F} \setminus (\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_i)$ so that $\nu(f_i, f_j) = 0$, $\forall i \neq j = n_i + 1, \ldots, n_{i+1}$. Again, this is up to renumbering. In this way, we exhaust $\mathcal{F}$ with a list $\mathcal{F}_1, \ldots, \mathcal{F}_N$ of such subsets. The sets $\mathcal{F}_1, \ldots, \mathcal{F}_N$ satisfy the conditions of Lemma 9.2, so

$$\# F_i \leq 2g - 2$$

for each $i$.

Because each $F_i$ is maximal in what is left over when we take away all previous sets, each $f \in F_j$ intersects some $f' \in F_j$, when $i > j$. More precisely, for each $i > j$, and for each $f_k, \in F_i$ there is an index $k_i(j)$ so that $f_{k_i(j)} \in F_j$ and

$$\nu(f_{k_i}, f_{k_i(j)}) \geq 1$$

Moreover, $\nu(f_{k_i}, f_{k_i}) \geq 1$ by assumption, so we can let $k_i(i) = i$ and get the above inequality for $j = i$, as well.

Thus,

$$\sum_{i=1}^{N} \sum_{f_k, \in F_i} \sum_{j=1}^{i} \nu(f_{k_i}, f_{k_i(j)}) \leq \frac{1}{2} \sum_{i, j} \nu(f_i, f_j)$$

because the left-hand some takes only some pairs of $f_i$ and $f_j$, and the right-hand sum takes every pair twice. As $\nu(f_{k_i}, f_{k_i(j)}) \geq 1$ for each $i \geq j$, we get

$$\sum_{i=1}^{N} \sum_{f_k, \in F_i} \sum_{j=1}^{i} 1 \leq \frac{1}{2} \sum_{i, j} \nu(f_i, f_j)$$

We can rework the left-hand side, and use that $\frac{1}{2} \sum_{i, j} \nu(f_i, f_j) \leq 16 \nu(\gamma, \gamma)$ to get

$$\sum_{i=1}^{N} \# F_i \leq 16 \nu(\gamma, \gamma)$$

Since $\# F_i \geq 1$ and $\nu(\gamma, \gamma) \leq K$, we get

$$\frac{N(N - 1)}{2} \leq 16K$$

Thus, for some universal constant $c'$, we get that

$$N \leq c' \sqrt{K}$$
(For example \( c' \) can be taken smaller than 8.)

We wish to bound \( m = \# F \). But

\[
\# F = \# F_1 + \cdots + \# F_N
\]

As \( \# F_i \leq 2g - 2 \) for each \( i \), and \( N \leq c' \sqrt{K} \), we get

\[
m \leq c \sqrt{K}
\]

for \( c = (2g - 2)c' \). Note that \( c \) depends only on the topology of \( S \). \( \square \)

9.3. **Bound on number of simple arcs in terms of length and intersection number.** We are done with the proof of Lemma 9.1 as soon as we bound the length of the sequence \( \gamma = \delta_1 \cdots \delta_m \) in terms of \( l_0(\gamma) \). Each \( \delta_i \) consists of at least one saddle connection. Let \( l_0 \) be the length of the shortest closed geodesic on \( X_0 \). Then \( l_0(\gamma) \geq l_0 m \). If \( l_0(\gamma) \leq L \), then

\[
m \leq \frac{L}{l_0}
\]

So, we have shown that

\[
m \leq \min\{L, c \sqrt{K}\}
\]

where \( c \) depends just on the topology of \( S \), and \( l_0 \) depends on the geometry of \( X_0 \). \( \square \)

10. **Size of \( G_c^*(L, K) \)**

We have shown that any \( \gamma \in G_c^*(L, K) \) can be written as \( \gamma = \delta_1 \cdots \delta_m \) for \( \delta_i \in C_0 \) for each \( i \), and \( m \leq \min\{\frac{L}{l_0}, c \sqrt{K}\} \) (Lemma 9.1). Furthermore, if \( l_0(\gamma) \leq L \), then \( l_0(\delta_i) \leq L \) for each \( i \). We have shown that \( \# C_0(L) \leq c_0 L^c \) where \( c_0 \) depends only on \( X_0 \) and \( c_g \) depends only on \( S \) (Lemma 8.1). The number of possible sequences \( \delta_1, \ldots, \delta_i \) that satisfy these criteria is at most

\[
(c_0 L^c)^{\min\{\frac{L}{l_0}, c \sqrt{K}\}}
\]

Furthermore, each sequence \( \delta_1, \ldots, \delta_m, \delta_i \in C_0 \) determines a closed geodesic. Thus the map from geodesics \( \gamma \) to ordered sets \( \delta_1, \ldots, \delta_m \) is injective. So the number of all possible sequences \( \delta_1, \ldots, \delta_m \) for which \( m \leq \min\{\frac{L}{l_0}, c \sqrt{K}\} \) and \( l_0(\delta_i) \leq L, \forall i \) bounds \( \# G_c^*(L, K) \) from above. Therefore,

\[
\# G_c^*(L, K) \leq \min\left\{\left(\frac{c_0 L^c}{l_0}\right)^{\frac{L}{l_0}}, \left(\frac{c_0 L^c}{c \sqrt{K}}\right)^{c \sqrt{K}}\right\}
\]
References


