LOWER BOUND FOR THE NUMBER OF NON-SIMPLE GEODESICS ON A PAIR OF PANTS

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1. INTRODUCTION

Let $S$ be a surface and let $P$ be a pair of pants. Geodesics on surfaces, and on pairs of pants specifically, have been studied extensively over the years. In this paper, we focus on getting a lower bound on the number of closed geodesics on $P$ with given upper bounds on length and self-intersection number. As a direct consequence, we get a lower bound for the number of such closed geodesics on any surface $S$.

In future papers, we will get upper bounds on this number for an arbitrary surface, and tighter upper bounds on pairs of pants $P$. The reason that the upper bounds are better on pairs of pants is that all geodesics there can be constructed explicitly, while geodesics on surfaces cannot.

1.1. Previous results for surfaces. There was a lot of work done in the 70’s and 80’s on counting closed geodesics on a negatively curved surface $S$ from the point of view of counting closed orbits of the geodesic flow. Let $\mathcal{G}^c$ be the set of closed geodesics on $S$ and let

$$\mathcal{G}^c(L) = \{ \gamma \in \mathcal{G}^c \mid l(\gamma) \leq L \}$$

where $l(\gamma)$ is the length of $\gamma$. The famous result in Margulis’s thesis states that if $S$ is negatively curved with a complete, finite volume metric, then

$$(1.1.1) \quad \#\mathcal{G}^c(L) \sim \frac{e^{\delta L}}{\delta L}$$

where $\delta$ is the topological entropy of the geodesic flow, and where $f(L) \sim g(L)$ if $\lim_{L \to \infty} \frac{f(L)}{g(L)} = 1$ [Mar70]. (Note that $\delta = 1$ when $S$ is hyperbolic.) A version of this result for hyperbolic surfaces was first proven by Huber [Hub59]. There are also many other, later versions of this result for non-closed surfaces. For example, see [CdV85, Pat88, LP82, Lal89] and [Gui86].

Recently, there has been work on the dependence of the number of closed geodesics on their self-intersection number as well as length. If $i(\gamma, \gamma)$ denotes the transverse self-intersection number of $\gamma \in \mathcal{G}^c$, let

$$\mathcal{G}^c(L, K) = \{ \gamma \in \mathcal{G}^c \mid l(\gamma) \leq L, i(\gamma, \gamma) \leq K \}$$

Then we can pose the following question.

**Question 1.** If $K = K(L)$ is a function of $L$, what is the asymptotic growth of $\#\mathcal{G}^c(L, K)$ in terms of $L$?

As part of her thesis, Mirzakhani showed that for a hyperbolic surface $S$ of genus $g$ with $n$ punctures,

$$\#\mathcal{G}^c(L, 0) \sim c(S)L^{6g-6+2n}$$
where \( c(S) \) is a constant depending only on the geometry of \( S \) [Mir08]. Rivin extended this result to geodesics with at most one self-intersection, to get that
\[
\#\mathcal{G}^c(L, 1) \sim c'(S)L^{6g-6+2n}
\]
where \( c'(S) \) is another constant depending only on the geometry of \( S \) [Riv12].

For arbitrary functions \( K = K(L) \), no asymptotic bounds are yet known. We can instead ask the following question as a first step to finding asymptotics.

**Question 2.** Given arbitrary \( L \) and \( K \), what are the best upper and lower bounds we can get on \( \#\mathcal{G}^c(L, K) \)?

Trivial bounds come from the fact that \( \#\mathcal{G}^c(L, 0) \leq \#\mathcal{G}^c(L, K) \leq \#\mathcal{G}^c(L) \), but these bounds do not have any dependence on the intersection number bound, \( K \). A first bound that does depend on \( K \) can be achieved as follows. The mapping class group of \( \text{Mod}_{g,n} \), acts on the set of closed geodesics and preserves self-intersection number. So consider the set of \( \text{Mod}_{g,n} \) orbits of geodesics. Let \( f(K) \) be the number of \( \text{Mod}_{g,n} \) orbits of closed geodesics with at most \( K \) self intersections. Just as there are finitely many topological types of simple closed curves, the number \( f(K) \) is finite. Then by combining the asymptotic result in [ABEM12] with results in [Bas13, Ker80, Wol79], we can get that for large \( L \) and arbitrary \( K \),
\[
\#\mathcal{G}^c(L, K) \sim f(K)L^{6g-6+2n}
\]
where \( A \) is the (finite) number of mapping class group orbits in \( \mathcal{G}^c(L, K) \). We write \( A \asymp B \) if there is a constant \( c \) depending only on the geometry of \( S \) s.t. \( \frac{1}{c}B \leq A \leq cB \). However, \( f(K) \) is not known for most \( K \). Our goal is to get bounds that are explicit in both length \( L \) and intersection number \( K \).

### 1.2. What we show.
In this paper, we get the following lower bound for a pair of pants \( P \):

**Theorem 1.1.** Let \( P \) be a pair of pants. Let \( l_{\text{max}} \) be the length of the longest boundary component of \( P \) or perpendicular geodesic arc connecting boundaries of \( P \). If \( L \geq 8l_{\text{max}} \) and \( K \geq 12 \), we have that
\[
\#\mathcal{G}^c(L, K) \geq \frac{1}{12} \min\{2\sqrt[6]{l_{\text{max}}}L, 2\sqrt[6]{K}\}
\]

This theorem implies the following result for an arbitrary surface:

**Theorem 1.2.** Let \( S \) be a genus \( g \) surface with \( n \) geodesic boundary components, and let \( X \) be a negatively curved metric on \( S \). Then whenever \( K > 12 \) and \( L > 3s_X \sqrt{K} \) we have
\[
\#\mathcal{G}^c(L, K) \geq c_X \left( \frac{L}{3\sqrt{K}} - s_X \right)^{6g-6+2n} 2^{\frac{3}{4n}}
\]
where \( s_X \) and \( c_X \) are constants that depend only on the metric \( X \).

As \( L \) goes to infinity, this theorem suggests a way to interpolate between the case when \( K \) is a constant and the case when \( K \) grows like \( L^2 \). If \( K \) is a constant, and \( L \) is very large, this theorem says \( \#\mathcal{G}(L, K) \geq c'_X L^{6g-6+2n} \), for \( c'_X \) a new constant depending on \( X \). This is consistent with the asymptotic results in [Mir08, Riv12] when \( K = 0 \) and 1. For \( K = O(L^2) \), however, we have that \( \frac{1}{\sqrt{K}} = O(1) \), and Theorem 1.2 gives an exponential lower bound on \( \#\mathcal{G}^c(L, K) \) in \( L \) that is consistent with Margulis’s result [Mar70]. This theorem demonstrates the transition from
polynomial to exponential growth of the number of geodesics on $S$ in terms of their length and self-intersection number.

1.3. Idea of proof. Theorem 1.1 is proven as follows:

- In Section 2, we create a combinatorial model for geodesics on a pair of pants. Each geodesic $\gamma$ can be represented as a cyclic word $w(\gamma)$ in a finite alphabet (Lemma 2.2). We then give some basic properties of these words in Section 2.2.
  These words are also the key ingredient in getting an upper bound on $\#G^c(L, K)$. This is done in a subsequent paper.

- We show that $l(\gamma) \leq |w|$ where $|w|$ denotes word length (Lemma 2.8), and $i(\gamma, \gamma) \leq i(w, w)$ (Lemma 2.11), where $i(w, w)$ is an intersection number for words defined in Definition 2.10.

- Then in Section 3, we construct a set of distinct geodesics. We show that each geodesic $\gamma$ arising from this construction lies in $G^c(L, K)$ by bounding $|w(\gamma)|$ and $i(w(\gamma), w(\gamma))$ from above.
  We get a lower bound on the number of geodesics we construct, giving us a lower bound on $\#G^c(L, K)$. For a more detailed summary, see Section 3.1.

We prove Theorem 1.2 in Section 4.
2. A COMBINATORIAL MODEL FOR GEODESICS ON PAIRS OF PANTS

Let \( \mathcal{P} \) be a hyperbolic pair of pants with geodesic boundary. In this section, we construct a combinatorial model for closed geodesics on \( \mathcal{P} \), and show that this model allows us to recover geometric properties of the corresponding geodesics. We do this as follows. First, there is a unique way to write \( \mathcal{P} \) as the union of two congruent right-angled hexagons. Take this decomposition (Figure 1).

![Figure 1. The hexagon decomposition of \( \mathcal{P} \) with boundary edges \( x_1 \) and \( x_2 \) and seam edge \( y_1 \) labeled.](image)

Let \( \mathcal{E} \) be the set consisting of two copies of each edge in the hexagon decomposition, one copy for each orientation. The set \( \mathcal{E} \) consists of oriented edges \( x_1, \ldots, x_{12} \) that lie on the boundary of \( \mathcal{P} \) and oriented edges \( y_1, \ldots, y_6 \) that pass through the interior of \( \mathcal{P} \). We call \( x_1, \ldots, x_{12} \) boundary edges and \( y_1, \ldots, y_6 \) seam edges.

We can model closed geodesics on \( \mathcal{P} \) by looking at closed concatenations of edges in \( \mathcal{E} \). If \( p \) is a closed concatenation of edges in \( \mathcal{E} \), then it corresponds to a cyclic word \( w \) with letters in \( \mathcal{E} \). We want to look at the following subset of such words.

**Definition 2.1.** Let \( \mathcal{W} \) be the set of cyclic words \( w \) with letters in \( \mathcal{E} \) so that
- The letters of \( w \) can be concatenated (in the order in which they appear) into a closed path \( p \).
- The curve \( p \) does not back-track.
- Lastly, we want a technical condition: each \( w \in \mathcal{W} \) can be written as \( w = b_1 s_1 \ldots b_n s_n \) where \( b_i \) is a sequence of boundary edges, \( |b_i| \geq 2 \), and \( s_i \) is a seam edge (\( |s_i| = 1 \)) for each \( i \), unless \( n = 1 \), in which case \( w = b_1 \).

Clearly, there is a map \( \mathcal{W} \to \mathcal{G}^c \). For each \( w \in \mathcal{W} \), we simply take the corresponding closed curve \( p(w) \). Each closed curve on \( \mathcal{P} \) has exactly one geodesic in its free homotopy class. Let \( \gamma(w) \) be the geodesic in the free homotopy class of \( p(w) \). Then the map \( w \mapsto \gamma(w) \) is well-defined.

2.1. **Turning Geodesics into Words.** Conversely, we can explicitly construct a map going back. In fact, we can construct an injective map \( \mathcal{G}^c \to \mathcal{W} \) sending each geodesic \( \gamma \) to some preferred word in \( \mathcal{W} \).

2.1.1. **The Projection** \( p(\gamma) \) of a Closed Geodesic \( \gamma \). For each closed geodesic \( \gamma \), we first construct a closed curve \( p(\gamma) \) that lies on the boundaries of the hexagons and is freely homotopic to \( \gamma \). Then \( p(\gamma) \) will be known as the projection of \( \gamma \) to the edges in \( \mathcal{E} \). We give the desired properties of \( p(\gamma) \) in the following lemma. These properties will allow us to convert \( p(\gamma) \) into a word \( w(\gamma) \in \mathcal{W} \).

**Lemma 2.2** (Construction of \( p(\gamma) \)). Let \( \gamma \) be a closed geodesic on \( \mathcal{P} \). Then there is a closed curve \( p(\gamma) \) that has the following properties:
(1) \( p(\gamma) \) is freely homotopic to \( \gamma \).
(2) \( p(\gamma) \) is a concatenation of edges in \( E \).
(3) each boundary edge in \( p(\gamma) \) is concatenated to at least one other boundary edge.

**Proof.** Let \( \gamma \) be an oriented, closed geodesic. The idea of the construction is given in the following four steps. See Figure 2 for the accompanying illustration.

(1) We break \( \gamma \) up into segments, which are pieces of \( \gamma \) that live entirely inside hexagons.
(2) Each segment \( \sigma \) lying inside a hexagon \( h \) gets projected to a sub-arc \( p'(\sigma) \) of the boundary of \( h \) (Figure 3). We can concatenate the arcs \( p'(\sigma) \) to get a closed curve \( p'(\gamma) \), which lies entirely in the boundaries of the two hexagons and is homotopic to \( \gamma \). At this stage, \( p'(\gamma) \) need not be the concatenation of edges in \( E \).
(3) We define a homotopy called Move 1 that we apply to finitely many disjoint sections of \( p'(\gamma) \). The result is a curve \( p''(\gamma) \) that is the concatenation of edges in \( E \).
(4) Lastly, we force each boundary edge to be concatenated to another boundary edge via a homotopy called Move 2, which we apply to sections of \( p''(\gamma) \). This gives us a closed curve \( p(\gamma) \) satisfying Lemma 2.2.

Now we fill in the details. Let a segment \( \sigma \) of \( \gamma \) be a maximal sub-arc that lies entirely in some hexagon \( h \) of the hexagon decomposition of \( P \). The projection \( p'(\sigma) \) of \( \sigma \) is the shortest sub-arc of the boundary of \( h \) that has the same endpoints
as \( \sigma \) and contains exactly one boundary edge. If the boundary edge is \( x \), we will say that \( \sigma \) is projected onto \( x \) (Figure 3.)

Since \( \sigma \) and \( p'(\sigma) \) have the same endpoints, and since \( \gamma \) is the concatenation of segments, we can concatenate all of the arcs \( p'(\sigma) \) into a closed curve \( p'(\gamma) \). By construction, \( p'(\gamma) \) is homotopic to \( \gamma \).

**Figure 3.** Projecting a segment \( \sigma \) onto the boundary edge \( x \).

**Move 1:** The goal is to homotope \( p'(\gamma) \) into a curve \( p''(\gamma) \) that is the concatenation of edges in \( E \). This is needed in the situation in Figure 4.

**Figure 4.** Move 1 is needed when \( \sigma \) and \( \sigma' \) project onto the same boundary component of \( P \).

**Figure 5.** When \( \sigma \) and \( \sigma' \) project onto different boundary components of \( P \), Move 1 is not needed.

The orientation of \( \gamma \) gives a cyclic ordering to its segments. Suppose segment \( \sigma \) is followed by segment \( \sigma' \). Write their projections as \( p'(\sigma) = \hat{y}_1 \circ x \circ \hat{y}_2 \) and \( p'(\sigma') = \hat{y}'_2 \circ x' \circ \hat{y}'_3 \), where \( x \) and \( x' \) are boundary edges and \( \hat{y}_i, \hat{y}'_i \) are pieces of the seam edge \( y_i \) for each \( i = 1, 2, 3 \). Note that because \( \sigma \) and \( \sigma' \) are consecutive segments, \( \hat{y}_2 \) and \( \hat{y}'_2 \) are both pieces of the same seam edge \( y_2 \). Furthermore, the endpoint of \( \hat{y}_2 \) is the start point of \( \hat{y}'_2 \). Thus the the concatenation \( \hat{y}_2 \circ \hat{y}'_2 \) is either null-homotopic (Figure 4) or it is all of \( y_2 \) (Figure 5.)

Move 1 is to homotope away concatenations of the form \( \hat{y} \circ \hat{y}' \) when they are null-homotopic. We apply it finitely many times to \( p'(\gamma) \) to get a new closed curve \( p''(\gamma) \) that is a concatenation of edges in \( E \). In fact, the number of times we must apply Move 1 is at most the number of segments in \( \gamma \). Note that \( p''(\gamma) \) is still homotopic to \( \gamma \).
Figure 6. Move 2. The boundary edge $x_1$ on the left is isolated, while the boundary edges $x_2$ and $x_3$ on the right are not.

**Move 2:** Now we homotope $p''(\gamma)$ to a new curve $p(\gamma)$ in which each boundary edge is concatenated to another boundary edge (Figure 6.) If a boundary edge $x$ is not concatenated to any other boundary edge, we call $x$ an **isolated boundary edge**.

Note that $p'(\gamma)$ never has more than 3 consecutive edges lying on the boundary of the same hexagon. Let $p'_1$ be any closed concatenation of edges in $\mathcal{E}$ with this property. We claim that we can homotope $p'_1$ to a curve $p''_2$ with strictly fewer isolated boundary edges.

Two boundary edges can be concatenated together (with no back-tracking) if and only if they lie on the boundaries of different hexagons. If $p'_1$ has an isolated boundary edge $x_1$, then it has a subarc of the form $y_2 \circ x_1 \circ y_3$ lying on the boundary of a single hexagon $h$, where $y_2$ and $y_3$ are seam edges. We will homotope it relative its endpoints to the other part of the boundary of $h$, which is an arc of the form $x_3 \circ y_1 \circ x_2$, where $x_3$ and $x_2$ are boundary edges and $y_1$ is a seam edge:

$$y_2 \circ x_1 \circ y_3 \mapsto x_3 \circ y_1 \circ x_2$$

This is **Move 2** (Figure 6). It gives us a new arc $p''_2$.

We claim that $p''_2$ has at least one fewer isolated boundary edge than $p'_1$. This is the same as showing that $x_2$ and $x_3$ are not isolated in $p''_2$. We have that $p'_1$ never follows more than three consecutive sides of a hexagon at a time. So $y_2$ and $y_3$ must be concatenated in $p'_1$ to edges in the other hexagon. These can only be the boundary edges $x'_2$ and $x'_3$ which lie on the same boundary components as $x_2$ and $x_3$, respectively. Thus $x_2$ and $x_3$ are not isolated.

Therefore, $p''_2$ has strictly fewer isolated boundary edges than $p'_1$. Since $p''(\gamma)$ has finitely many (isolated) boundary edges, we can perform Move 2 finitely many times to get a closed curve $p(\gamma)$ with no isolated boundary edges.

**Remark 2.3.** Applying Move 2 can get rid of more than one isolated boundary edge at a time. Thus the final arc $p(\gamma)$ depends on the order in which we get rid of isolated boundary edges. For each closed geodesic $\gamma$, we make a choice of $p(\gamma)$ once and for all.

2.1.2. **Defining the Cyclic Word $w(\gamma)$ for a Closed Geodesic $\gamma$**. By Remark 2.3, we force the map $\gamma \mapsto p(\gamma)$ to be well-defined. Since $p(\gamma)$ is a concatenation of edges in $\mathcal{E}$, it corresponds to a cyclic word $w(\gamma)$ with letters in $\mathcal{E}$. Thus each $\gamma \in \mathcal{G}^c$ corresponds to a unique word $w(\gamma)$. 

We show in Lemma 2.5 that \( w(\gamma) \in \mathcal{W} \), where \( \mathcal{W} \) is the set defined in Definition 2.1. Let

\[
\mathcal{W}(L, K) = \{ w(\gamma) \in \mathcal{W} \mid \gamma \in \mathcal{G}^c(L, K) \}
\]

2.1.3. \textit{Injective Correspondence for Closed Geodesics}. The most important relationship between closed geodesics in \( \mathcal{G}^c \) and words in \( \mathcal{W} \) is that distinct geodesics correspond to different words. This is because \( \gamma \) is homotopic to \( p(\gamma) \), so if two geodesics correspond to the same word, they are homotopic as well. Since there is exactly one geodesic in each free homotopy class of non-trivial closed curves, the map from closed geodesics to words is injective. We formalize this in the following remark.

\textbf{Remark 2.4.} If \( \gamma \neq \gamma' \) are two distinct closed geodesics on \( \mathcal{P} \) then \( w(\gamma) \neq w(\gamma') \).

2.2. \textbf{Word Structure: Boundary Subwords}. We want to examine the form of a word \( w(\gamma) \) in more detail. A closed geodesic \( \gamma \in \mathcal{G}^c \) spends most of its time twisting about boundary components of \( \mathcal{P} \). So its projection \( p(\gamma) \) spends most of its time winding around those boundary components. Note that to transition from one boundary component to another, \( p(\gamma) \) only needs to take a single seam edge (Figure 7). Thus, \( w(\gamma) \) has long sequences of boundary edges (called boundary subwords) separated by single seam edges (Lemma 2.5.) Furthermore, \( w(\gamma) \) is completely determined by the sequence of boundary subwords that appear (Lemma 2.6.) This is because there is at most one seam edge connecting one boundary edge to another.

\textbf{Figure 7.} An approximation to \( p(\gamma) \)

\textbf{Lemma 2.5.} For each \( \gamma \in \mathcal{G}^c \), \( w(\gamma) \in \mathcal{W} \), where \( \mathcal{W} \) is the set defined in Definition 2.1.
Proof. Let $\gamma \in \mathcal{G}^\circ$. By construction, $w(\gamma)$ can be concatenated into a closed curve $p(\gamma)$, and this curve $p(\gamma)$ does not back-track. We just need to show that

$$w(\gamma) = b_1s_1\ldots b_ns_n$$

where $b_i$ consists only of boundary edges, with $|b_i| \geq 2$ and $s_i$ is a single seam edge for each $i$, unless $n = 1$, in which case $w(\gamma) = b_1$.

In Section 2.1.2, we noted that $w(\gamma)$ can always be written to start with a boundary edge and end with a seam edge, unless $w(\gamma)$ consists only of boundary edges. (By construction, $w(\gamma)$ always contains at least 1 boundary edge.) So we can always write

$$w(\gamma) = b_1s_1\ldots b_ns_n$$

where $b_i$ is a non-empty sequence of boundary edges, and where $s_i$ is a non-empty sequence of seam edges for each $i$, unless $w(\gamma) = b_1$.

Let $n \geq 2$ and let $p(\gamma)$ be the curve corresponding to the word $w(\gamma)$. By construction, each boundary edge in $p(\gamma)$ is concatenated to another boundary edge. Therefore $|b_i| \geq 2$ for each $i$. We constructed $p(\gamma)$ to never back-track. Thus, each seam edge can only be concatenated to a boundary edge. Therefore, $|s_i| = 1$ for each $i$.

If $n = 1$, we want to show that $|b_1| \geq 2$ and $s_1$ is an empty word. Take the curve $p(\gamma)$ corresponding to $w(\gamma)$. The subarc of $p(\gamma)$ corresponding to $b_1$ lies on a single boundary component of $\mathcal{P}$. Seam edges have endpoints on distinct boundary components of $\mathcal{P}$. So if $p(\gamma)$ had a seam edge, its start and endpoints would be on different boundary components, and it would not be closed. So $s_1$ must be empty. The condition that each boundary edge is concatenated to another boundary edge guarantees that $|b_1| \geq 2$.

We now get a few more properties of the structure of $w(\gamma)$.

**Lemma 2.6.** A cyclic word $w(\gamma)$ is completely determined by its sequence $b_1, \ldots, b_n$ of boundary subwords. That is, if $w(\gamma) = b_1s_1\ldots b_ns_n$ and $w(\gamma') = b_1s'_1\ldots b_ns'_n$ (where the boundary subwords of $w$ and $w'$ are the same and in the same order), then $s_i = s'_i$ for each $i$.

**Proof.** This lemma follows from the fact that any two boundary subwords can be joined together by at most one seam edge. In particular, given two oriented boundary edges $x$ and $x'$ that lie on different boundary components of $\mathcal{P}$, there is at most one oriented seam edge $y$ such that we can concatenate them into an oriented arc $x \circ y \circ x'$. We assume that $w$ and $w'$ come from closed geodesics $\gamma$ and $\gamma'$. If $x$ is the last boundary edge in the subword $b_i$ and $x'$ is the first boundary edge in the subword $b_i$ then the existence of $p(\gamma)$ implies that there does exist a seam edge $y$ such that $x \circ y \circ x'$ is an oriented arc. Thus $s_i = s'_i$ for each $i$. (Note that we number the boundary subwords modulo $n$, so $b_{n+1} = b_1$.)

Lastly, we note that because each $\gamma \in \mathcal{G}^\circ$ is primitive, so is the word $w(\gamma)$.

**Lemma 2.7.** Let $w' = w(\gamma')$. Suppose there exists a word $w$ in the edges in $\mathcal{E}$ such that

$$w' = w^n$$

for $n > 1$. Then $\gamma'$ is not primitive.
Proof. Suppose \( w = b_1s_1 \ldots b_ns_n \) where all the subwords except for possibly \( s_n \) are non-empty. Then the fact that \( w' = b_1s_1 \ldots b_n s_n b_1 \ldots \) implies that the concatenation \( b_n \circ s_n \circ b_1 \) corresponds to an oriented path in the edges of \( E \). Thus we can concatenate the edges in \( w \) into a closed path \( p \). If \( p' \) is the closed curve corresponding to \( w' \), then \( p' = p^n \). Every closed curve has a unique closed geodesic in its free homotopy class. Let \( \gamma \) be the geodesic in the free homotopy class of \( p \). Then \( p' = p^n \) implies that \( \gamma' = \gamma^n \). Therefore \( \gamma' \) is not primitive. \( \square 

2.3. Word and Geodesic Lengths. The word \( w(\gamma) \) encodes geometric properties of \( \gamma \) for each \( \gamma \in \mathcal{G}^c \). For example, we get the following relationship between the length of a closed geodesic \( \gamma \) and the word length of \( w(\gamma) \).

Lemma 2.8. Let \( \gamma \in \mathcal{G}^c \). If \(|w(\gamma)|\) is the word length of \( w(\gamma) \), then

\[
\frac{1}{3} l_{\min}|w(\gamma)| \leq l(\gamma) \leq l_{\max}|w(\gamma)|
\]

where \( l_{\min} \) is the length of the shortest boundary edge, and \( l_{\max} \) is the length of the longest boundary or seam edge in \( E \).

Proof. Let \( \gamma \in \mathcal{G}^c \) and let \( w(\gamma) \in \mathcal{W} \) be the associated word. Let \( p(\gamma) \) be the closed curve corresponding to \( w(\gamma) \). Throughout this proof, we will use that the number of edges in \( p(\gamma) \) is exactly the word length of \( w(\gamma) \).

To get the upper bound, we use that \( l(\gamma) \leq l(p(\gamma)) \) since a geodesic is the shortest curve in its free homotopy class. Thus, if \( l_{\max} \) is the length of the longest edge in \( E \), then

\[
l(\gamma) \leq l_{\max}|w(\gamma)|
\]

To get the lower bound, set \( n \) to be the number of segments in \( \gamma \). We first compare \( n \) to \(|w(\gamma)|\). Let \( m \) to be the number of boundary edges in \( p(\gamma) \). Note that \( m \leq 2n \). To see this, look at the construction of \( p(\gamma) \) in the proof of Lemma 2.2. Each segment \( \sigma \) of \( \gamma \) got projected onto a single boundary edge, which then may have been replaced by two boundary edges when we did Move 2. Thus, the boundary edges in \( p(\gamma) \) account for at most two times the number of segments in \( \gamma \).

Since \( p(\gamma) \) does not backtrack, two seam edges can never be concatenated together. Also, we know that boundary edges appear in consecutive pairs. Thus, as least \( \frac{2}{3} \) of all edges in \( p(\gamma) \) are boundary edges. In other words, \( \frac{2}{3}|w(\gamma)| \leq m \). Therefore,

\[
\frac{2}{6}|w(\gamma)| \leq \frac{1}{2}m \leq n
\]

where \( \gamma \) has \( n \) segments and \( p(\gamma) \) has \( m \) boundary edges.

Suppose a segment \( \sigma \) has endpoints on seam edges \( y \) and \( y' \). Because we broke \( \mathcal{P} \) up into right angle hexagons, \( y \) and \( y' \) meet a common boundary edge \( x \) at right angles. By some hyperbolic geometry, any arc connecting \( y \) and \( y' \) will thus be at least as long as \( x \), i.e. \( l(\sigma) \geq l(x) \). Thus, if \( l_{\min} \) is the length of the smallest boundary edge in \( E \), then \( l_{\min} \leq l(\sigma) \) for each segment \( \sigma \). Therefore, \( l_{\min}n \leq l(\gamma) \). So we get the lower bound:

\[
\frac{1}{3}l_{\min}|w(\gamma)| \leq l(\gamma)
\]
2.4. An intersection number for words. We want a notion of word self-intersection number so that if \( i(w(\gamma), w(\gamma)) \leq K \), then \( i(\gamma, \gamma) \leq K \).

**Definition 2.9.** Let \( b_i \) be a boundary subword of \( w \in \mathcal{W} \). Suppose \( \beta \) is a boundary component of \( \mathcal{P} \). We write \( b_i \subset \beta \) and say that \( b_i \) lies on \( \beta \) if the boundary edges in \( b_i \) lie on \( \beta \).

We define the self-intersection number of a word as follows.

**Definition 2.10.** Let \( w = b_1s_1 \ldots b_ns_n \in \mathcal{W} \). Let \( \beta_1, \beta_2 \) and \( \beta_3 \) be the boundary components of \( \mathcal{P} \). Suppose \( w \) has \( n_j \) boundary subwords lying on \( \beta_j \), \( j = 1, 2, 3 \). Let \( \sigma_j : \{1, \ldots, n_j\} \to \{1, \ldots, n\} \) so that

- \( b_{\sigma_j(1)} \ldots b_{\sigma_j(n_j)} \subset \beta_j \)
- \( |b_{\sigma_j(1)}| \geq |b_{\sigma_j(2)}| \geq \cdots \geq |b_{\sigma_j(n_j)}| \)

Then, let

\[
i(w, w) = 2 \sum_{j=1,2,3} \sum_{i=1}^{n_j} i(b_{\sigma_j(i)})\]

In other words, given a word \( w = b_1s_1 \ldots b_ns_n \), we group the boundary subwords \( b_1, \ldots, b_n \) according to the component of \( \partial \mathcal{P} \) on which they lie, and then we order the boundary subwords in each group from largest to smallest word length. This gives us the reindexing functions \( \sigma_j \), \( j = 1, 2, 3 \). Then we form the sum above.

For example, suppose

\[
w = b_1s_1 \ldots b_5s_5 \in \mathcal{W}\]

Suppose \( b_1, b_3, b_5 \subset \beta_1 \) and \( b_2, b_4 \subset \beta_2 \). Suppose further that \( |b_5| \geq |b_1| \geq |b_3| \) and \( |b_2| \geq |b_4| \). Then

\[
i(w, w) = 2|b_3| + 2|b_1| + 3|b_5| + 2\left[|b_2| + 2|b_4|\right]\]

**Lemma 2.11.** Suppose \( w \in \mathcal{W} \) corresponds to the geodesic \( \gamma \in \mathcal{G}^c \). Then

\[
i(\gamma, \gamma) \leq i(w, w)\]

**Proof.** Suppose \( w = b_1s_1 \ldots b_ns_n \) is a cyclic word that corresponds to some closed geodesic \( \gamma \). We will show how to construct a closed curve \( \delta \) freely homotopic to \( \gamma \) where

\[
|\delta \cap \delta| \leq i(w, w)\]

Let \( \beta_1, \beta_2, \beta_3 \) be the boundary components of \( \mathcal{P} \). Suppose \( w \) has \( n_j \) boundary subwords lying on \( \beta_j \), \( j = 1, 2, 3 \). Let \( \sigma_j : \{1, \ldots, n_j\} \to \{1, \ldots, n\} \) so that

- \( b_{\sigma_j(1)} \ldots b_{\sigma_j(n_j)} \subset \beta_j \)
- \( |b_{\sigma_j(1)}| \geq |b_{\sigma_j(2)}| \geq \cdots \geq |b_{\sigma_j(n_j)}| \)

First, we construct a region of \( \mathcal{P} \) homotopic to the one skeleton of its hexagon decomposition (Figure 8). Let \( R_1, R_2 \) and \( R_3 \) be disjoint neighborhoods of the three side edges. For each boundary subword \( b_{\sigma_j(i)} \) lying on \( \beta_j \), let \( C^j_i \) be a cylinder embedded in \( \mathcal{P} \) that is homotopic to \( \beta_j \). Choose these cylinders so that any two cylinders \( C^j_i \) and \( C^j_k \) are pairwise disjoint. Lastly, choose \( C^j_i \) to be closer to \( \beta_j \) than \( C^j_{i+1} \) for each \( i, j \). The union of \( R_1 \cup R_2 \cup R_3 \) and the cylinders is homotopic in \( \mathcal{P} \) to the union of hexagon edges. (See Figure 8).

We now construct \( \delta \) so that

\[
\delta \subset \bigcup_{k=1,2,3} \bigcup_{i,j} R_k \cup C^j_i
\]
Figure 8. The curve $\delta$ will lie in the gray regions.

Given a cylinder $C_j^i$, we say its top is the boundary component closest to $\beta_j$ and its bottom is the other boundary component. If $b_{\sigma_j(i)} s_{\sigma_j(i)} b_{\sigma_l(k)}$ is a subword of $w$, we draw a line $l_j^i$ from the top of $C_j^i$ to the bottom of $C_k^l$. We draw $l_j^i$ inside the unique region $R_m$, $m \in \{1, 2, 3\}$ that connects the two cylinders. We require that all of the lines in the set $\{l_j^i\}$ be pairwise disjoint.

Let $p_j^i$ be the endpoint of $l_j^i$ on cylinder $C_j^i$, and let $q_k^l$ be the endpoint of $l_k^l$ on cylinder $C_k^l$. Since $\gamma$ is closed, each cylinder $C_j^i$ is now decorated with a point $p_j^i$ on its top boundary and a point $q_k^l$ on its bottom boundary. The boundary subword $b_{\sigma_j(i)}$ determines a twisting direction about $\beta_j$. So in each cylinder $C_j^i$, we draw a curve from $p_j^i$ to $q_k^l$ that twists in this direction for $|b_{\sigma_j(i)}|$ half-twists. Call this curve $\delta_j^i$.

There is just one natural way to concatenate the twisting arcs $\delta_j^i$ with the lines $l_j^i$ to form a closed curve. Call this concatenation $\delta$. So,

$$\delta = \bigcup_{i,j=1}^n \delta_j^i \bigcup_{i,j=1}^n l_j^i$$

where we take the concatenations in the order that makes sense (Figure 9).

Now we just need to count the self-intersections of $\delta$. Since $R_1, R_2$ and $R_3$ are pairwise disjoint, and since any two pairs of cylinders are also disjoint, the only intersections occur when a curve $\delta_j^i$ in a cylinder $C_j^i$ intersects a line $l_k^l$ in a region $R_k$. If $l_k^l$ passes through $C_j^i$, and if $\delta_j^i$ has $|b_{\sigma_j(i)}|$ half-twists, then

$$|l_k^l \cap \delta_j^i| \leq \frac{|b_{\sigma_j(i)}|}{2} + 1$$

The curve $\delta_j^i$ is intersected only by lines with endpoints on the cylinders $C_1^j, \ldots, C_l^j$. In particular, both lines coming out of $C_1^j, \ldots, C_{i-1}^j$ cross $\delta_j^i$. However, $\delta_j^i$ is only
Figure 9. An example of a curve $\delta$ constructed from a word $w$.

interacted by the line with endpoint on the top boundary component of $C^j_i$, not the line whose endpoint is on the bottom boundary. Thus, each $C^j_i$ contains at most $(2i - 1)(\frac{1}{2}|b_{\sigma_j(i)}| + 1)$ unique intersection points of $\delta \cap \delta$. Therefore,

$$|\delta \cap \delta| \leq \sum_{j=1,2,3} \sum_{i=1}^{n_j} i(|b_{\sigma_j(i)}| + 2)$$

since $(2i - 1)((\frac{1}{2}|b_{\sigma_j(i)}| + 1) = (|b_{\sigma_j(i)}| + 2)(i - 1)$ and $i - 1 \leq i$.

Since geodesics have the least number of self-intersections in their free homotopy class, this implies that

$$i(\gamma, \gamma) \leq \sum_{j=1,2,3} \sum_{i=1}^{n_j} i(|b_{\sigma_j(i)}| + 2)$$

Lastly, since $|b_i| \geq 2$ for each $i = 1, \ldots, n$, we have that $|b_{\sigma_j(i)}| + 2 \leq 2|b_{\sigma_j(i)}|$. Therefore, we arrive at

$$i(\gamma, \gamma) \leq \sum_{j=1,2,3} \sum_{i=1}^{n_j} 2i|b_{\sigma_j(i)}|$$

$\square$
3. Constructing geodesics in $G^c(L, K)$ to get a lower bound

In this section, we prove the lower bound on $\#G^c(L, K)$ for a pair of pants.

**Theorem 3.1** (Theorem 1.1). Let $P$ be a pair of pants. Let $l_{\text{max}}$ be the length of the longest closed geodesic in $P$ or geodesic arc with endpoints on $\partial P$, perpendicular to $\partial P$. If $L \geq 8l_{\text{max}}$ and $K \geq 12$, we have that

$$\#G^c(L, K) \geq \frac{1}{12} \min\{2 \frac{l_{\text{max}}}{32L}, 2 \frac{\sqrt{K}}{16}\}$$

3.1. **Proof summary.** The proof of the theorem is organized as follows.

- We have an injection $G^c \rightarrow W$, and there is a map $W \rightarrow G^c$ back. Unfortunately, the latter map is only surjective, since the map $G^c \rightarrow W$ required us to make choices to make it well-defined.

  We get a partial inverse to the map $G^c \rightarrow W$. In Lemma 3.2, we give a construction that turns closed paths in the graph $\Gamma_E$, found in Figure 11, into words that lie in $W$. This means we get a map from closed paths $\tau$ in $\Gamma_E$ to closed geodesics $\gamma(\tau) \in G^c$:

  $$\tau \rightarrow w(\tau) \rightarrow \gamma(\tau)$$

  In Lemma 3.6, we show that this map is one-to-one. So we get an injection from words in $W$ that come from closed paths in $\Gamma_E$ to closed geodesics in $G^c$.

- The construction that takes closed paths in $\Gamma_E$ to geodesics is simple enough that path length gives an upper bound on geodesic length and self-intersection number. We find a function $N(L, K)$ so that if $\tau$ is a closed path in $\Gamma_E$ and $\gamma(\tau)$ is the corresponding closed geodesic then

  $$|\tau| \leq N(L, K) \implies \gamma(\tau) \in G^c(L, K)$$

  where $|\tau|$ is the path length of $\tau$.

  To find $N(L, K)$, we use the map $\tau \rightarrow w(\tau) \in W$ as an intermediary. The construction of $w(\tau)$ directly implies that $3|\tau| \geq |w(\tau)|$ and $18|\tau| \geq \sqrt{i(w, w)}$. We then apply Lemmas 2.8 and 2.11 to get the relationship between $|\tau|$ and the quantities $i(\gamma(\tau))$ and $i(\gamma(\tau), \gamma(\tau))$.

- Then we get a lower bound on the number of closed paths in $\Gamma_E$ of length $N$ (Lemma 3.8). Actually, we only do this for $N \equiv 2 \mod 6$ and $n \geq 8$, but this is enough.

- Lastly, we put everything together to get a lower bound on the number of closed geodesics on $P$ in Section 3.6.

3.2. **Building words that correspond to closed geodesics.** Consider the labeling of the (oriented) edges in $E$ given in Figure 10. If $x_i$ is a labeled edge, then the same edge with the opposite orientation will be denoted $x_i^{-1}$.

**Lemma 3.2.** Any closed path $\tau$ in the directed graph $\Gamma_E$ in Figure 11 corresponds to a cyclic word $w(\tau) \in W$ and an oriented closed geodesic $\gamma(\tau)$.

**Example 1.** The path $x_1x_2x_4^{-1}x_3^{-1}$ corresponds to the cyclic word

$$w = (x_1x_4^{-1}x_1) \cdot s_1 \cdot (x_2x_6^{-1}) \cdot s_2 \cdot (x_4^{-1}x_1x_4^{-1}) \cdot s_3 \cdot (x_3^{-1}x_6) \cdot s_4$$

where $s_1, \ldots, s_4$ are the unique side edges that make $w$ a word in $W$. 
Proof. We start with a description of \( \Gamma_E \). The vertices of \( \Gamma_E \) are the edges in \( E \) as labeled in Figure 10. The vertices on the front hexagon are labeled by edges \( x_1, \ldots, x_6 \) and the vertices on the back hexagon are labeled by edges \( x_1^{-1}, \ldots, x_6^{-1} \). The edges of \( \Gamma_E \) are directed. For each \( i \), an edge labeled \( o \) connects vertex \( x_i \) to \( x_{i+1} \) and \( x_i^{-1} \) to \( x_{i-1}^{-1} \). An edge labeled \( e \) connects \( x_i \) to \( x_{i+2}^{-1} \).

Take a closed path \( \tau \) in \( \Gamma_E \) with \( \tau = v_1 \cdots v_n \) (where \( v_i \) is a vertex of \( \Gamma_E \) for each \( i \)). We associate to each vertex \( v_i \) a boundary subword \( b_i \). The first letter of \( b_i \) is the edge label of \( v_i \). If \( v_i \) is joined to \( v_{i+1} \) by an edge labeled \( o \), then \( |b_i| = 3 \). Otherwise \( |b_i| = 2 \). Note that specifying the initial letter and length of the boundary subword uniquely determines \( b_i \).

In Example 1, \( v_1 \) has label \( x_1 \) and \( x_1 \) is joined to \( x_2 \) by an edge labeled \( o \). So \( b_1 = x_1 x_4^{-1} x_1 \) has length 3 and starts with \( x_1 \).

We claim that there are seam edges \( s_1, \ldots, s_n \) so that \( b_1 s_1 \cdots b_n s_n \in \mathcal{W} \). There are four cases to check: \( v_i \) could have the form \( x_j \) or \( x_j^{-1} \) and the edge from \( v_i \) to \( v_{i+1} \) could be labeled either \( e \) or \( o \). Suppose \( v_i \) is labeled \( x_j \) and the edge from \( v_i \) to \( v_{i+1} \) is labeled \( e \), so \( |b_i| = 2 \). Then given the edge labels in Figure 10, \( b_i = x_j x_{j+3}^{-1} \). The edge labeled \( e \) joins vertex \( x_j \) to vertex \( x_{j+2}^{-1} \). Thus, \( v_{i+1} = x_{j+2}^{-1} \). So \( b_{i+1} \) starts with \( x_{j+2}^{-1} \). There is a seam edge between \( x_{j+3}^{-1} \) and \( x_{j+2}^{-1} \) for all \( j = 1, \ldots, 6 \). So there is a seam edge \( s_i \) so that \( b_i s_i b_{i+1} \) forms a non-backtracking path. The other cases can be checked in the same way.

Therefore, the path \( \tau = v_1 \cdots v_n \) corresponds to a word \( w(\tau) = b_1 s_1 \cdots b_n s_n \in \mathcal{W} \). Since each word in \( \mathcal{W} \) corresponds to a closed geodesic, \( \tau \) corresponds to a closed geodesic \( \gamma(\tau) \in \mathcal{G}^c \).

\( \square \)
3.3. **Map from paths in \( \Gamma_E \) to geodesics injective.** We now have maps
\[
\tau \rightarrow w(\tau) \rightarrow \gamma(\tau)
\]
We show that \( w(\tau) \) actually lies in the following special class of words.

**Definition 3.3.** A word \( b_1 s_1 \ldots b_n s_n \in \mathcal{W} \) is a **cyclic alternating word** if no two consecutive boundary edges lie on the same hexagon. In particular, the last edge of \( b_i \) does not lie in the same hexagon as the first edge of \( b_{i+1} \).

**Claim 3.4.** For each closed path \( \tau \) in \( \Gamma_E \), the word \( w(\tau) \) constructed in the proof of Lemma 3.2 is a cyclic alternating word.

**Proof.** Let \( \tau = v_1 v_2 \ldots v_n \) be a closed path in \( \Gamma_E \). Then \( \tau \) corresponds to a word \( b_1 s_1 \ldots b_n s_n \in \mathcal{W} \). On each boundary subword, adjacent boundary edges lie on different hexagons. So we just need to check that the last boundary edge of \( b_i \) lies on a different hexagon than the first boundary edge in \( b_{i+1} \). Once again, this can be done by considering four cases. We have the cases where \( v_i \) is of the form \( x_j \) or the form \( x_j^{-1} \) and the cases where \( v_i \) is joined to \( v_{i+1} \) by an edge labeled \( e \) or an edge labeled \( o \).

We will do the case where \( v_i \) is labeled \( x_j \) and \( v_i \) is joined to \( v_{i+1} \) by an edge labeled \( e \). If \( v_i \) is labeled \( x_j \) and the edge is labeled \( e \), then \( b_i = x_j x_{j+1}^{-1} \). Since \( v_i \) is joined to \( v_{i+1} \) by an edge labeled \( e \), we have that \( b_{i+1} \) starts with the letter \( x_{j+1}^{-1} \).

We see from Figure 10 that \( x_{j+3}^{-1} \) and \( x_{j+2}^{-1} \) lie on different hexagons. The other 3 cases are shown in the same way. \( \square \)

**Definition 3.5.** We say a path is cyclic if it is a cyclic word in its vertices. It is primitive if the word is primitive.

**Lemma 3.6.** Suppose \( \tau, \tau' \) are two closed, primitive cyclic paths in \( \Gamma_E \). Then \( \gamma(\tau) \neq \gamma(\tau') \).

**Proof.** Let \( w(\tau) \) and \( w(\tau') \) be the words constructed from \( \tau \) and \( \tau' \), respectively. We know that \( w(\tau) \) and \( w(\tau') \) are cyclic alternating words. Because \( \tau \) and \( \tau' \) are primitive, \( w(\tau) \) and \( w(\tau') \) are primitive as well. We will show that if \( w \) and \( w' \) are primitive, cyclic alternating words, then their geodesics \( \gamma(w) \) and \( \gamma(w') \) are distinct.

Let \( w \) be a primitive, cyclic alternating word and let \( \gamma(w) \) be the corresponding geodesic. Let \( p(w) \) be the arc in \( P \) formed by concatenating the edges in \( w \). Lift \( p(w) \) to a complete closed curve \( \tilde{p}(w) \) in the universal cover, \( \hat{P} \).

The hexagon decomposition of \( P \) lifts to a hexagonal tiling of \( \hat{P} \). We get a graph \( \Gamma \) dual to this hexagonal tiling: Put a vertex in the middle of each hexagon, and join two vertices if their hexagons share a side edge. This graph is a valence 3 tree. (See Figure 12).

Let \( \Gamma(w) \) be the subgraph of \( \Gamma \) that has a vertex for every hexagon that contains a boundary edge of \( \tilde{p}(w) \) (see Figure 13). We want to show that \( \Gamma(w) \) is an embedded line. If not, then \( \Gamma(w) \) would have a valence 1 vertex. This would correspond to \( \tilde{p}(w) \) entering a hexagon \( h \), traversing some of its boundary edges, and then leaving \( h \) through the same seam edge through which it entered. But this cannot be achieved if \( \tilde{p}(w) \) never has more than one consecutive boundary edge in the same hexagon. Thus \( \Gamma(w) \) is an embedded line.

We can do the same construction for any other primitive, cyclic alternating word \( w' \). Let \( \gamma(w') \) be the geodesic corresponding to \( w' \) and let \( p(w') \) be the concatenation
Figure 12. A piece of the graph $\Gamma$ dual to the hexagonal tiling of $\tilde{\mathcal{P}}$

Figure 13. The subgraph $\Gamma(w)$ goes through the same hexagons as $\tilde{\gamma}(w)$.

of edges in $w'$. Lift $p(w')$ to a curve $\tilde{p}(w')$ and construct the subgraph $\Gamma(w')$. This subgraph is a line embedded in $\Gamma$.

Note that complete geodesics in $\tilde{\mathcal{P}}$ are in one-to-one correspondence with embedded lines in $\Gamma$. Therefore, $\Gamma(w)$ corresponds to a unique complete geodesic $\tilde{\gamma}$ that must be a lift of $\gamma(w)$, and likewise, $\Gamma(w')$ corresponds to a unique complete geodesic $\tilde{\gamma}'$ that must be a lift of $\gamma(w')$ (see Figure 13).

Suppose for contradiction that $\gamma(w) = \gamma(w')$ as oriented geodesics. So we could have chosen a lift $\tilde{p}(w')$ so that $\Gamma(w) = \Gamma(w')$ as oriented paths in $\Gamma$. (Note that the orientations on $\Gamma(w)$ and $\Gamma(w')$ come from the orientations of $p(w)$ and $p(w')$, respectively.) The vertices of $\Gamma(w)$ are in one-to-one correspondence with the hexagons traversed by $\tilde{p}(w)$, and the analogous statement is true for $\tilde{p}(w')$. So $\tilde{p}(w)$ and $\tilde{p}(w')$ pass through the exact same hexagons in $\tilde{\mathcal{P}}$.

Suppose $\Gamma(w)$ and $\Gamma(w')$ pass through consecutive hexagons $h_1, h_2$ and $h_3$. Then $\tilde{p}(w)$ and $\tilde{p}(w')$ both travel from $h_1$ to $h_3$ through the boundaries of these hexagons. Furthermore, they each pass through just one boundary edge in $h_2$. Therefore, they
both pass through the same boundary edge of \( h_2 \) (see Figure 14.) Thus, \( \tilde{p}(w) \) and \( \tilde{p}(w') \) pass through all the same boundary edges. But there is just one side edge that can lie between a pair of boundary edges. So \( \tilde{p}(w) \) and \( \tilde{p}(w') \) must be equal as paths. Since \( w \) and \( w' \) are primitive, this implies \( p(w) = p(w') \). So \( w = w' \) as cyclic words.

Figure 14. If \( \tilde{p}(w) \) and \( \tilde{p}(w') \) pass through \( h_1, h_2 \) and \( h_3 \), and \( w \) and \( w' \) are cyclic alternating words, then \( \tilde{p}(w) \) and \( \tilde{p}(w') \) must pass through the same boundary edge of \( h_2 \).

We have that \( w(\tau) \) and \( w(\tau') \) are cyclic alternating words, and we suppose that \( \gamma(\tau) = \gamma(\tau') \) as oriented geodesics. Thus \( w(\tau) = w(\tau') \) as cyclic words. Since \( w(\tau) \) always has more than one boundary subword, we can recover \( \tau \) from \( w(\tau) \). In other words, the map \( \tau \to w(\tau) \) is injective. So \( \tau = \tau' \) as cyclic paths. Therefore two primitive, cyclic paths \( \tau \) and \( \tau' \) in \( \Gamma_E \) are equal if and only if \( \gamma(\tau) = \gamma(\tau') \) as oriented geodesics.

\[ \square \]

3.4. **Path length and geodesic length and intersection number.** Fix \( L \) and \( K \). We want to count the number of closed paths \( \tau \) in \( \Gamma_E \) so that the corresponding closed geodesic satisfies \( \gamma(\tau) \in G_c^c(L, K) \). First, we show that we can guarantee \( \gamma(\tau) \in G_c^c(L, K) \) with just an upper bound on path length \( |\tau| \).

**Lemma 3.7.** Fix \( L \) and \( K \). Suppose we have a closed path \( \tau \) in \( \Gamma_E \) of length at most \( N(L, K) \), for

\[ N(L, K) = \min\left\{ \frac{1}{4l_{\max}}L, \sqrt{\frac{L}{3}K} \right\} \]

Then \( \gamma(\tau) \in G_c^c(L, K) \), where \( \gamma(\tau) \) is the closed geodesic corresponding to \( \tau \).

**Proof.** Take a closed path \( \tau \) in \( \Gamma_E \) of length at most \( N(L, K) \). Then \( \tau \) corresponds to a word \( w(\tau) = b_1s_1 \ldots b_ns_n \). By construction,

\[ n = |\tau| \]

where \( |\tau| \) denotes the path length of \( \tau \).
We constructed $w(\tau)$ so that $|b_i| \leq 3$ for each $i$. Thus, $|b_is_i| \leq 4, \forall i$. Therefore,

$|w(\tau)| \leq 4|\tau|$

Furthermore, we can get a bound on $i(w(\tau), w(\tau))$ as follows. We have that

$$i(w(\tau), w(\tau)) = 2 \sum_{j=1,2,3} \sum_{i=1}^{n_j} i|b_{\sigma(j)}|$$

for appropriate indices $\sigma_1(i), \sigma_2(i)$ and $\sigma_3(i)$. Recall that $n_j$ is the number of $b_i$ that lie on boundary component $\beta_j$ of $P$. Thus,

$$i(w(\tau), w(\tau)) \leq 2 \sum_{j=1,2,3} \sum_{i=1}^{n_j} 3i$$

$$= 3[(n_1^2 + n_1) + (n_2^2 + n_2) + (n_3^2 + n_3)]$$

$$\leq 3(n_1 + n_2 + n_3)^2$$

$$= 3n^2 = 3|\tau|^2$$

Note that we get the second inequality because $n_1, n_2$ and $n_3$ are positive integers. Therefore,

$$i(w(\tau), w(\tau)) \leq 3|\tau|^2$$

Let $\gamma(\tau)$ be the geodesic corresponding to $\tau$ (and $w(\tau)$). By Lemmas 2.8 and 2.11,

$$l(\gamma(\tau)) \leq l_{max}|w(\tau)|$$

and

$$i(\gamma(\tau), \gamma(\tau)) \leq i(w(\tau), w(\tau))$$

where $l_{max}$ is the length of the longest boundary or seam edge in $E$. Therefore,

$$l(\gamma(\tau)) \leq 4l_{max}|\tau|$$

and

$$i(\gamma(\tau), \gamma(\tau)) \leq 3|\tau|^2$$

In particular, if

$$|\tau| \leq \frac{1}{4l_{max}}L$$

and

$$|\tau| \leq \sqrt{\frac{K}{3}}$$

then $\gamma(\tau) \in G^c(L, K)$.

\[\square\]

3.5. Counting paths. By Lemma 3.7, we can get a lower bound on $G^c(L, K)$ via a lower bound on the number of paths of length $N(L, K)$. The following lemma gives us this lower bound for special $N$. We restrict the numbers $N$ we consider to make the computations simpler, but the proof can be adjusted to work for all $N$.

Lemma 3.8. Suppose $N \equiv 2 \mod 6$. The number of non-cyclic paths of length $N$ in $\Gamma_E$ that start and end at $x_1$ is exactly

$$\frac{1}{3}(2^{N-1} + 1)$$

Proof. We prove this lemma by constructing the adjacency matrix $M$ for the graph $\Gamma_E$. The $i^{th}$ row (or column) corresponds to vertex $x_i$ if $1 \leq i \leq 6$ and to vertex $x_{i-6}^{-1}$ if $7 \leq i \leq 12$. We put a 1 in position $(i, j)$ if there is a direct edge from the vertex corresponding to row $i$ to the vertex corresponding to column $j$. The $(1,1)$
entry of $M^N$ tells us how many paths there are of length $N$ from $x_1$ back to itself. We can compute that

$$M = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

where $A$ is a matrix such that $A^6 = I$. We can show by induction that

$$M^N = \begin{bmatrix}
A^{2-N}(A^2 + I)^{N-1} & A^{3-N}(A^2 + I)^{N-1} \\
A^{-N-1}(A^2 + I)^{N-1} & A^{-N}(A^2 + I)^{N-1}
\end{bmatrix}$$

Let $a_{11}$ be the $(1,1)$ entry of $M^N$. In particular, $a_{11}$ is the $(1,1)$ term of $A^{2-N}(A^2 + I)^{N-1}$. If $N \equiv 2 \pmod{6}$, then $A^6 = I$ implies $a_{11}$ is the $(1,1)$ term of

$$(A^2 + I)^{N-1} = \sum_{i=0}^{N-1} \binom{N-1}{i} A^{2i}$$

The $(1,1)$ entry of $A^{2i}$ is 1 if and only if $2i \equiv 0 \pmod{6}$. In other words, it is 1 if and only if $i \equiv 0 \pmod{3}$. In other words,

$$a_{11} = \sum_{i \equiv 0 \pmod{3}} \binom{N-1}{i}$$

It is well-known that

$$\sum_{i \equiv 0 \pmod{3}} \binom{N-1}{i} = \frac{1}{3} \left(2^{N-1} + (-\omega^2)^{N-1} + (-\omega)^{N-1}\right)$$

where $\omega$ is the primitive $3^{rd}$ root of unity. We have that $N \equiv 2 \pmod{6}$, so $N - 1$ is odd and $N - 1 \equiv 1 \pmod{3}$. As $\omega^3 = 1$, we have that

$$(-\omega)^{N-1} = -\omega \text{ and } (-\omega^2)^{N-1} = -\omega^2$$

Lastly, $-\omega - \omega^2 = 1$ as $1 + \omega + \omega^2 = 0$. Therefore,

$$a_{11} = \frac{1}{3}(2^{N-1} + 1)$$

In other words, there are exactly $\frac{1}{3}(2^{N-1} + 1)$ paths from $x_1$ back to itself of length $N$, where $N \equiv 2 \pmod{6}$. \qed
3.6. **Proof of Theorem 1.1.** We now get a lower bound on $\#G^c(L, K)$. By Lemma 3.7, if $\tau$ is a closed path in $\Gamma_\mathcal{E}$ with $|\tau| \leq N(L, K)$ for

$$N(L, K) = \min\left\{ \frac{1}{4l_{\text{max}}} L, \sqrt{\frac{K}{3}} \right\}$$

then $\gamma(\tau) \in G^c(L, K)$. So we need to get a lower bound on the number of primitive, cyclic paths in $\Gamma_\mathcal{E}$ of length at most $N(L, K)$.

Suppose $N(L, K) < 2$. There are no closed paths in $\Gamma_\mathcal{E}$ of length less than 2, so in this case we should get a lower bound of 0. Assume $N(L, K) \geq 2$. In that case, there is some $N \in \mathbb{N}$ so that

$$\frac{1}{4} N(L, K) \leq N \leq N(L, K)$$

and $N \equiv 2 \mod 6$.

By Lemma 3.8, there are exactly $\frac{1}{3}(2^{N-1} + 1)$ distinct paths of length $N$ in $\Gamma_\mathcal{E}$ that start and end at $x_1$. Since each word has length $N$, there are at least $\frac{1}{3N}(2^{N-1} + 1)$ cyclic paths of length exactly $N$ in $\Gamma_\mathcal{E}$. (In fact, each word gets counted once for each appearance of $x_1$, after being reduced to a primitive word. So $v_1v_2v_1v_2$ would get counted once, but $v_1v_2v_3v_2$ would get counted twice.) This is therefore a lower bound on the number of primitive cyclic paths of length at most $N$.

Note that

$$\frac{1}{3n}(2^{n-1} + 1) \geq \frac{1}{12} 2^{\frac{n}{2}}$$

for all $n > 0$. To see this, note that this is equivalent to the statement $\frac{4}{n}(2^{\frac{n}{2}} - 1 + 2^{\frac{n}{2}}) \geq 1$. We examine the derivative of the expression on the right hand side of the inequality. The derivative is equal to zero when

$$\frac{n + 2}{n - 2} = 2^{n-1}$$

This equation has a solution somewhere between 3 and 4. As $\frac{4}{n}(2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}})$ goes to infinity as $n$ goes to 0 or $\infty$, its derivative is zero at its minimum. If $3 < n < 4$, then both terms of $\frac{4}{n}(2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}})$ are greater than 1. So $\frac{4}{n}(2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}}) \geq 1$ for all $n$.

As $N(L, K) \geq 2$, there is some $M$ with $\frac{N(L, K)}{4} \leq M \leq N(L, K)$ and $M \equiv 2 \mod 6$. Thus

$$\#G^c(L, K) \geq \frac{1}{12}(2^{\frac{M}{2}}) \geq \frac{1}{12}(2^{\frac{N(L, K)}{8}})$$

We have that $N(L, K) \geq 2$ whenever $L \geq 8l_{\text{max}}$ and $K \geq 12$. Therefore, if $L \geq 8l_{\text{max}}$ and $K \geq 12$, we have that

$$\#G^c(L, K) \geq \min\{2^{\frac{1}{8l_{\text{max}}} L}, 2^{\frac{\sqrt{K}}{8}} \}$$
Let $S$ be an arbitrary surface. The lower bound for $G^c(L, K)$ on $S$ follows from the lower bound on pairs of pants. The idea is that we will count geodesics in different pairs of pants inside $S$. To make sure that we do not over-count, we need the following lemma.

**Lemma 4.1.** Let $f_1, f_2 : \mathcal{P} \to S$ be two homeomorphisms of a pair of pants into $S$. Suppose $\gamma_1$ and $\gamma_2$ are two non-simple closed curves on $\mathcal{P}$. If $f_1$ is not homotopic to $f_2$, then $f_1(\gamma_1)$ is not freely homotopic to $f_2(\gamma_2)$.

**Proof.** Suppose $f_1(\gamma_1)$ is freely homotopic to $f_2(\gamma_2)$. Then the geodesic representatives of $f_1(\gamma_1)$ and $f_2(\gamma_2)$ are the same. Let $\phi$ be the geodesic representative of these two curves.

Let $\alpha_1, \alpha_2$ be two boundary curves of $\mathcal{P}$. Any curve in $\mathcal{P}$ can be written in $\pi_1(\mathcal{P})$ as words in $\alpha_1, \alpha_2$ and their inverses. Let $\rho$ be a two-petaled rose in $\mathcal{P}$ whose two petals are freely homotopic to $\alpha_1$ and $\alpha_2$. Thus any non-simple closed curve in $\mathcal{P}$ is freely homotopic to a curve lying entirely on $\rho$. Therefore, $\phi$ is freely homotopic to $f_1(\rho)$ and to $f_2(\rho)$. In other words, $f_1(\rho)$ is freely homotopic to $f_2(\rho)$. But $f_1(\mathcal{P})$ and $f_2(\mathcal{P})$ deformation retract onto $f_1(\rho)$ and $f_2(\rho)$, respectively. So $f_1$ is homotopic to $f_2$, which is a contradiction. 

We will now prove the main theorem:

**Theorem 4.2.** Let $S$ be a genus $g$ surface with $n$ geodesic boundary components, and let $X$ be a negatively curved metric on $S$. Then whenever $K > 12$ and $L > 3s_X \sqrt{K}$ we have

$$\# G^c(L, K) \geq c_X \left( \frac{L}{3\sqrt{K}} - s_X \right) 6^{g-6+2n} 2^{\frac{K}{16}}$$

where $s_X$ and $c_X$ are constants that depend only on the metric $X$.

**NB:** The constant $s_X$ is the length of the longest arc connecting some boundary components of a pair of pants in $S$, at right angles.

**Proof.** Consider the set of all pairs of pants $\mathcal{P}$ with geodesic boundary components inside $S$. Given any such $\mathcal{P} \subset S$, let $l_{\max}(\mathcal{P})$ be the length of the longest boundary component or longest arc connecting boundaries of $\mathcal{P}$ at right angles. Then by Theorem ?? and Lemma 4.1

$$\# G^c(L, K) \geq \sum_{\mathcal{P} \subset S, \text{ } L \geq 8l_{\max}(\mathcal{P})} \min\left\{ \frac{1}{12}(2^{\frac{1}{16}l_{\max}(\mathcal{P})}L), \frac{1}{12}(2^{\frac{K}{16}}) \right\}$$

The condition $L \geq 8l_{\max}(\mathcal{P})$ on each pair of pants $\mathcal{P}$ in the above sum comes from Theorem ???. The other condition (that $K \geq 12$) is already assumed. Furthermore, simple closed curves are not counted by Theorem ??, so we do not have to worry about overcounting them.

Choose a pair of pants $\mathcal{P}$. Take any perpendicular arc $\alpha$ connecting the boundaries of $\mathcal{P}$. The length of this arc is controlled by the widths of the collar neighborhoods of $\partial \mathcal{P}$. If the collar neighborhoods have widths $w_1, w_2$ and $w_3$ then $l(\alpha) \leq w_1 + w_2 + w_3$. The collar neighborhoods get wider as the boundary curves get shorter. $(S, X_{-1})$ has some shortest simple closed curve. So there is some
constant $s_X$ depending only on $X$ so that if $\alpha$ is the longest perpendicular arc connecting any two boundary components of a pair of pants, then $l(\alpha) \leq s_X$.

Thus, for any pair of pants $P \subset \mathcal{S}$, we have that

$$l_{\text{max}}(P) \leq l(P) + s_X$$

where $l(P)$ is the sum of the lengths of the boundary components of $P$.

Fix $L$ and $K$. Let $l_0 = \frac{L}{3\sqrt{K}}$. Then whenever $l_{\text{max}} < l_0$ we have

$$2^{\frac{1}{l_{\text{max}}}} > 2^{\frac{X}{16}}$$

Thus, we can simplify the lower bound on $\#G^c(L, K)$ as follows:

$$\#G^c(L, K) \geq \sum_{P \subset \mathcal{S}} \frac{1}{l_{\text{max}}(P)} 2^{\frac{X}{16}}$$

Note that we must also have $L > 8l_{\text{max}}(P)$ for all $P$ in the sum. However, if $K \geq 12$, then $L \geq 8l_0$, so this condition holds automatically.

By [Mir08], the number of pairs of pants $P$ with length at most $L$ grows asymptotically like $c(X)L^{6g-6+2n}$. So there is some constant $c'(X)$ so that this number is bounded below by $c'(X)L^{6g-6+2n}$ for all $L > 11$ (if $L \geq 10$, then $l_0 > 1$, so this is a reasonable condition). If $l(P) < l_0 - s_X$ then $l_{\text{max}}(P) < l_0$. The number of pairs of pants with $l(P) < l_0 - s_X$ is at most $c'(X)(l_0 - s_X)^{6g-6+2n}$ (whenever $l_0 > s_X$). So there are at least $c'(X)(l_0 - s_X)^{6g-6+2n}$ pairs of pants so that $l_{\text{max}}(P) < l_0$. So whenever $l_0 > s_X$ and $K > 12$, we have

$$\#G^c(L, K) \geq d(X)(l_0 - s_X)^{6g-6+2n}2^{\frac{X}{16}}$$

for some constant $d(X)$ depending only on the metric $X$.

Note that $l_0 > s_X$ means that $L > 3s_X\sqrt{K}$. \qed
References


