1 Previous Results

\( S \) - a surface
\( \mathcal{G} \) - set of closed geodesics
\( i(\gamma, \gamma) \) - geodesic length
\( i(\gamma, \gamma) \) - self-intersection number
\( \mathcal{G}'(L) = \{ \gamma \in \mathcal{G} \mid i(\gamma) \leq L \} \)

\[ \#(\mathcal{G}'(L)) = \frac{e^L}{L!} \]

Theorem 1 (Margulis) Let \( S \) be a closed, negatively curved surface. Then
\[ \#(\mathcal{G}'(L)) = \frac{e^L}{L!} \quad A \sim B \text{ if } \lim_{L \to \infty} \frac{A}{B} = 1 \]
where \( A \) is the topological entropy of the geodesic flow.

NB: If \( S \) is hyperbolic, then \( \delta = 1 \).

\[ \mathcal{G}'(L, K) = \{ \gamma \in \mathcal{G}' \mid i(\gamma) \leq L, i(\gamma, \gamma) \leq K \} \]

Question 1 Support \( K = K(L) \) is a function of \( L \). What can be said about the asymptotic growth of \( \#(\mathcal{G}'(L, K)) \)?

Theorem 2 (Mirzakhani) Let \( S \) be a hyperbolic genus \( g \) surface with \( n \) punctures. Then
\[ \#(\mathcal{G}'(L, K)) \sim c(S) L^{2g-2+n} \]
where \( c(S) \) is a constant depending only on the geometry of \( S \).

Theorem 3 (Rivin) Let \( S \) be a hyperbolic genus \( g \) surface with \( n \) punctures. Then
\[ \#(\mathcal{G}'(L, K)) \sim c(S) L^{2g-2+n} \]
where \( c(S) \) is a constant depending only on the geometry of \( S \).

Question 2 For fixed \( L \) and \( K \), what are the best bounds for \( \#(\mathcal{G}'(L, K)) \)?

NB: Trivial upper and lower bounds are \( 0 \leq \#(\mathcal{G}'(L, K)) \leq \#(\mathcal{G}'(L)) \).

Consequence of Athreya-Bubuï̧on-Eskin-Mirzakhani-Boaventura-Parlier-Baasmanjian + others

Proposition 1 On a hyperbolic genus \( g \) surface with \( n \) punctures,
\[ \#(\mathcal{G}'(L, K)) \leq f(K) L^{2g-2+n} \]
where \( f(K) \) is the number of \( \text{Mod}_g \) orbits in \( \mathcal{G}'(L, K) \).

A \sim B \text{ if } \frac{A}{B} \leq A \leq eB \]

2 Combinatorial Model on Arbitrary Surfaces

Reduce to pairs of pants.

1 Combining Points on Pairs of Pants

Pants as union of two hexagons.

2 Get word in hexagon edges.

\( \gamma \rightarrow w(\gamma) = b_1^{a_1} \cdots b_n^{a_n} \)

\( b_i \) on \( \mathcal{G} \), \( a_i \) is a corner of \( \mathcal{P} \).

\( \bullet i(\gamma, \gamma) \leq n(\gamma) \leq \sum i(\gamma) \leq \sum i(\gamma_i) \)

2 For Pairs of Pants

\( \mathcal{P} \) - hyperbolic pair of pants with geodesic boundary

Theorem (S) On a pair of pants \( \mathcal{P} \),
\[ e^{\sqrt{i(\gamma, \gamma)}} \leq \#(\mathcal{G}'(L, K)) \leq \min(\sqrt{i(\gamma, \gamma)} a, \sqrt{i(\gamma, \gamma)} b, \sqrt{i(\gamma, \gamma)} c) \]

where \( c \) depends on the geometry of \( \mathcal{P} \), and \( c \to 0 \) as the lengths of \( \partial \mathcal{P} \) go to infinity, and \( c \) is a universal constant.

Corollary (S) If \( K = K(L) \) is s.t. \( K = o(L^2) \), then
\[ \#(\mathcal{G}'(L, K)) = o(\#(\mathcal{G}'(L))) \]

Theorem (Lalley) Let \( S \) be a closed hyperbolic surface. Choose \( \gamma_2 \in \mathcal{G}'(L) \) at random for each \( L \in \mathbb{N} \). Then
\[ \lim_{L \to \infty} \frac{i(\gamma_1, \gamma_2)}{L^2} = \kappa \]

for almost any choice of sequence \( \gamma_2 \), where \( \kappa \) depends only on the geometry of \( S \).

Theorem 5 (Balasch) Let \( S \) be a hyperbolic surface. Then
\[ i(\gamma, \gamma) \leq M(\gamma)^2 \]

for any \( \gamma \in \mathcal{G}' \), where \( M \) depends only on the geometry of \( S \).

3 For an Arbitrary Surface

Conjecture (S) On an arbitrary surface \( S \),
\[ \#(\mathcal{G}'(L, K)) \leq \min(e^{L}, e^{K/2} \log \#(\mathcal{G}'(L, K))) \]

where \( \#(\mathcal{G}(L, K)) \) is a rational function in \( K \) and \( L \) and \( c \) is a constant depending only on the geometry of \( S \).

Question 2 For which families \( \mathcal{G}' \subset \mathcal{G} \) of complete geodesics with infinitely many self-intersections do the conclusions of Birman-Series hold?

\( \mathcal{G}' \) - set of complete geodesics
\( \mathcal{K} = \{ \gamma \in \mathcal{G} \mid i(\gamma, \gamma) \leq K \} \)
\( \mathcal{K}_x \) - points on some geodesic in \( \mathcal{K} \)

NB: Since most complete geodesics have infinitely many self-intersections, the geodesic in \( \mathcal{K} \) should be thought of as almost simple.

Theorem 1 (Birman-Series) Let \( S \) be a hyperbolic surface. Then \( \mathcal{K}_x \) is nowhere dense and has Hausdorff dimension 1.

1 The Original Theorem

Points on arbitrary closed geodesics. (Dense)

\( \mathcal{S}_x \) (Credit: Buser, Fanlari)

\[ \begin{align*}
\mathcal{G}_x &= \{ \gamma \in \mathcal{G} \mid i(\gamma, \gamma) < \epsilon \} \\
\mathcal{F}_x \subset \mathcal{S}_x \text{ - set of points on some } \gamma \in \mathcal{G}_x \\
\mathcal{F}_x \text{ - set of points on some } \gamma \in \mathcal{G}_x \\
\mathcal{F}_x \text{ - set of points on some } \gamma \in \mathcal{G}_x \\
\end{align*} \]

Theorem 2 (S) On \( \mathcal{P} \), \( \mathcal{F}_x \) has Hausdorff dimension \( \mu(\epsilon) \) where \( \lim_{\epsilon \to 0} \mu(\epsilon) = 1 \).

In particular, \( \mathcal{F}_x \) has Hausdorff dimension 1.

But, \( \mathcal{F}_x \) is not nowhere dense. In fact, it can have positive Lebesgue measure.

Proposition 3 (S) If \( \mathcal{F}_x \) denotes the closure of \( \mathcal{F}_x \) in \( \mathcal{P} \), then
\[ \mathcal{F}_x = \mathcal{F} \]

More regularity:
\[ \mathcal{G}_x(\epsilon) = \{ \gamma \in \mathcal{G} \mid i(\gamma, \gamma) < \epsilon \} \]

\[ \mathcal{F}_x(\epsilon) \text{ - set of points on some } \gamma \in \mathcal{G}_x(\epsilon) \]

Theorem 4 (S) There is an \( \epsilon_0 \) s.t. \( \forall \epsilon < \epsilon_0, \mathcal{F}_x(\epsilon) \) is nowhere dense for all \( \epsilon \).

3 For an Arbitrary Surface

Conjecture 1 On an arbitrary surface, the Birman-Series theorem holds when
\[ i(\gamma, \gamma) = o(L^{2/3}) \]