A) DO NOT OPEN YOUR EXAM UNTIL YOU ARE TOLD TO DO SO.

B) Be sure to write your name on this page and check your section code.

C) This is a closed book, closed notes exam. No calculators allowed.

D) No cell phones are allowed. Please turn them off during this exam.

E) No other electronic devices allowed. Same rules as cell phones.

F) Every exam is worth a total of 100 points. There are 6 problems on this exam, the first one is worth 10 points and the others are each worth 18 points. Each exam has 7 pages including the cover sheet. Check to see that you have all the pages.

G) Be sure to show all of your work in order to get complete credit.

H) If you finish early, please hand in your exam and leave quietly in consideration of your fellow students. You may need to show a picture ID with a clear picture when you turn in your exam. When time is up, you will be instructed to put down your writing utensil and close your exam.
(1) Consider a linear differential equation \( \frac{dy}{dx} = p(x)y(x) + g(x) \). State the existence and uniqueness theorem for this first order equation, giving complete assumptions about the coefficient functions \( p \) and \( g \) and properties of the solution function \( y \).

Assume \( p, g \) are continuous on an interval \( I = (a, b) \).

Then there exists a unique function \( y \) such that

\[
y' = py + g
\]

on all of \( I \). The function \( y \) has a continuous derivative.
(2) Consider the differential equation $y' = xy + 3x$.

a) What is the general solution to this equation?

b) What is the solution such that $y(0) = -1$?

c) What is the domain of the solution to b)?

\[ u(x) = e^{\int -x \, dx} = e^{-x^2/2} \]

The solution is:

\[ y(x) = \int e^{-x^2/2} 3x \, dx + C \]

\[ = \int e^{-x^2/2} \, e^{x^2/2} \]

\[ = -3 + Ce^{x^2/2} \]

b) \[-1 = y(0) = -3 + C \implies C = 2 \]

**Solution:** $y(x) = -3 + 2e^{x^2/2}$

c) Domain of $y$ is all of $\mathbb{R} = (-\infty, \infty)$
(3) Consider the differential equation \( \frac{dQ}{dt} = 100 - \frac{Q}{10} \).

a) Find the general solution of this differential equation and compute 
\( L = \lim_{t \to \infty} Q(t) \).

b) Is there a reason you could have guessed the value of \( L \) without solving the equation?

\[ \text{The integrating factor is} \]
\[ u(t) = e^{\int \frac{1}{10} \; dt} = e^{\frac{t}{10}} \]

\[ \text{The solution} \]
\[ y(t) = \frac{\int e^{\frac{t}{10}} \cdot 100 \; dt + C}{e^{\frac{t}{10}}} \]
\[ = 1000 + Ce^{-\frac{t}{10}} \]

\[ \lim_{t \to \infty} y(t) = 1000 \]

b) Setting \( \frac{dQ}{dt} = 0 \), we see
\[ 100 - \frac{Q}{10} = 0 \Rightarrow Q = 1000 \]

This "suggests" there is an equilibrium at \( Q = 1000 \)

(\text{Note: } Q = 1000 \text{ is a solution for} \)

(4) Consider the differential equation

$$\frac{dy}{dx} = \frac{(3x^2 \cos(xy) - yx^3 \sin(xy) + 2x) + (1 - x^4 \sin(xy))}{y}.$$

a) What is the general solution to this equation?

b) What is the solution such that \( y(\pi) = 1 \)?

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\[ \Phi(x, y) = \int N \, dy \]

\[ = \int (1 - x^4 \sin(xy)) \, dy \]

\[ = y + x^3 \cos(xy) + h(x) \]

\[ \frac{\partial \Phi}{\partial x} = 3x^2 \cos(xy) - x^3 y \sin(xy) + h'(x) \]

So \( h'(x) = 2x \).

This proves the equation is exact.

\[ \Phi(x, y) = y + x^3 \cos(xy) + x^2 \]

**SOLN:** \( C = \Phi(x, y) \)

b) \( C = \Phi(\pi, 1) = 1 + \pi^3 (-1) + \pi^2 \cdots 

**Implicit SOLN:** \( \Phi(x, y) = \pi^2 + 1 - \pi^3 \)
(5) Consider the differential equation \( \frac{dy}{dx} = \frac{x^2+xy+y^2}{2y} \).

a) Is it separable?
b) Is it exact?
c) Is it homogeneous?
d) What is the general solution to this equation?
e) What is the solution such that \( y(1) = 2 \)?

\[ d) \quad \frac{y}{x} = \frac{y}{x} \]

\[ \sqrt{1 + \frac{dy}{dx}} = \frac{dy}{dx} = 1 + y + y^2 \]

\[ = \int \frac{dy}{1+y^2} = \int \frac{dx}{x} \]

\[ = \arctan(y) = \ln|x| + C \]

\[ \Rightarrow \quad \frac{y}{x} = \tan(\ln|x| + c) \quad \text{implicit soln.} \]

\[ e) \quad y = \frac{2}{6} \tan(0 + c) \]

\[ \Rightarrow \quad c = \arctan(2) \]

So \( y = \frac{x \tan(\ln(x) + \tan^{-1}(2))}{6} \).
(6) Consider the logistic differential equation $y' = 2y(10 - y)$. What is the formula for the general solution of this differential equation?

$$\text{LHS} = \int \frac{dy}{y(10-y)} = \int 2dx = 2x + C$$

$$\text{LHS} = \int \frac{A}{y} + \frac{B}{10-y} dy$$

$\Rightarrow A = \frac{1}{10}$ and $B = \frac{1}{10}$

$$2x + C = \int \frac{1}{10} + \frac{1}{10-y} dy$$

$$20x + C = \ln |10/10-y| + C$$

$$20x + C = \ln |1/y|$$

$$Ke^{20x} = \left| \frac{y}{10-y} \right|$$

$$\frac{y}{10-y} = Ke^{20x}$$

$$y = 10Ke^{20x}/(1+Ke^{20x})$$

A *complet* solution.

Given $y(10) = y_0$, this could be solved explicitly.