Proper holomorphic mappings throughout mathematics

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Plan of talk:

- Proper holomorphic mappings
- Blaschke products, automorphisms
- Spherical versus homotopy equivalence
- The degree is not a homotopy invariant when $n \geq 2$
- Uncountably many spherical equivalence classes
- Finitely many homotopy classes for ratl. maps when $n \geq 2$
- Links with other parts of math
- Result of Baouendi-Huang and effect of subtle change in hypothesis
Let $X, Y$ be locally compact topological spaces and $f : X \to Y$ continuous. $f$ is called **proper** if $K$ compact in $Y$ implies $f^{-1}(K)$ compact in $X$.  

If $X$ and $Y$ are bounded domains in $\mathbb{C}^n$, then $f$ is proper if and only if $x_n \to b_X$ implies $f(x_n) \to b_Y$.  

Let $X^\ast, Y^\ast$ be the one-point compactifications. Define $F : X^\ast \to Y^\ast$ by $F(x) = f(x)$ and $F(\infty) = \infty$. Then a continuous $f : X \to Y$ is proper if and only if $F$ is continuous: This idea is crucial to CR Geometry.
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This idea is crucial to CR Geometry.
Let $X$ and $Y$ be bounded domains in $\mathbb{C}^n$, and assume $f : X \to Y$ is holomorphic and proper. Assume also that the boundaries are smooth. When $f$ has an extension to $bX$, the restriction $f_0$ maps $bX$ to $bY$. Thus $f_0$ is a CR mapping.

$f$ satisfies a first order system of PDE, called the tangential Cauchy-Riemann equations, together with a non-linear boundary condition.
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Smokey Robinson, in the 1967 song I second that emotion sang A taste of honey’s worse than none at all.
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Computing volumes of images leads to a variational problem for the complex Monge-Ampere determinant and an $L^2$ estimate.
Today we will emphasize comparing spherical equivalence with homotopy equivalence.

Two of the results are joint with Jiri Lebl.

First I want to introduce spherical equivalence and to motivate homotopy in this context.
Proper self-maps of balls:

**Theorem**
(almost known to Poincare. Proved independently by H. Alexander and Pincuk in the 1970’s) Assume $n = N \geq 2$. Assume $f : B_n \to B_n$ proper, holomorphic. Then $f$ is an automorphism.
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What is \( \text{Aut}(B_n) \)?

\[
U \frac{a - L_a(z)}{1 - \langle z, a \rangle}
\]

Here \( U \) is unitary, \( ||a|| < 1 \), and

\[
L_a(z) = \frac{\langle z, a \rangle a}{s + 1} + sz.
\]

Here \( s = \sqrt{1 - ||a||^2} \).

The automorphism group is transitive.
Theorem
A proper holomorphic self-map of the unit disk is a finite Blaschke product:

\[ f(z) = e^{i\theta} \prod_{j=1}^{m} \frac{z - a_j}{1 - \overline{a_j}z}. \]

The degree can be any natural number. We get a branched cover. There are \( m \) copies of the disk.
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When I first heard of the Alexander result (almost 40 years ago), I wondered “what happened to \( z^m? \)” but I was afraid to ask the speaker. (Dan Burns) Thinking about it has led me all over the place.
Replace each $a_j$ in the Blaschke product by $(1 - t)a_j$ and replace $\theta$ by $(1 - t)\theta$. We get a one-parameter family of Blaschke products $H_t$ with $H_0 = f$ and $H_1 = z^d$. Thus each proper map is \textbf{homotopic} to the map $z \rightarrow z^m$. Furthermore $m$ is unique.

$$m = \frac{1}{2\pi i} \int \frac{f'(z)}{f(z)}dz.$$ 

Conclude: When $n = N = 1$, homotopy classes correspond to natural numbers. We would get all integers if we just considered maps of the circle.
Assume $f, g : B_n \to B_N$ are proper holomorphic. They are \textbf{spherically equivalent} if there are automorphisms $\phi, \chi$ such that

$$f = \chi \circ g \circ \phi.$$  

They are \textbf{homotopy equivalent} if there is a one-parameter family of proper holomorphic maps $H_t$ such that $H_0 = f$ and $H_1 = g$. We assume the map $(z, t) \to H_t(z)$ is continuous.
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Consequence: If $H_t(z) = \sum c_\alpha(t)z^\alpha$, then each $c_\alpha$ is continuous.

Since $\|H_t(z)\|^2 = \sum \langle c_\alpha(t), c_\beta(t) \rangle z^\alpha \overline{z}^\beta$, the map to the space of Hermitian forms (with appropriate topology) is also continuous.
Spherical equivalence implies homotopy equivalence, converse is false.
We need two refinements of the definition of homotopy equivalence.
First, when \( f, g \) are rational we may wish to restrict to each \( H_t \) being rational.
Theorem of Forstneric: \( n \geq 2 \). If \( f \) proper between balls, and sufficiently smooth at the sphere, then \( f \) is rational (and, by Cima-Suffridge, extends holomorphically past.)
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Second. The target dimension matters.
Assume $f : B_n \to B_{N_1}$ and $g : B_n \to B_{N_2}$ are proper holomorphic.
They are **homotopic in target dimension** $N$ if there is a homotopy (of proper maps) between $f \oplus 0$ and $g \oplus 0$. 
Lemma

Any pair of proper holomorphic maps are homotopic in dimension $N = N_1 + N_2$. In fact, for $N = \max(N_1, N_2) + n$.

Proof.

Put $t = \cos(\theta)$. Consider

$$H_t = \cos(\theta)f \oplus \sin(\theta)g.$$ 

Then $H_t$ gives a homotopy into dimension $N_1 + N_2$. Being more clever we can lower the dimension. Apply it to $f$ and $g = z$ to get a homotopy in dimension $N_1 + n$. Apply to $z$ and $g$ to get a homotopy into dimension $n + N_2$. Then since homotopy is an equivalence relation we get the result.

Difficult? Given $f, g$ proper, find the minimum $N$ in which they are homotopic in target dimension $N$. 

Surprising fact. The degree is not a homotopy invariant.

**Example**

$f, g : \mathbf{B}_2 \rightarrow \mathbf{B}_5$. Each map has embedding dimension 5. These maps are of different degree but they are homotopic in target dimension 5.

\[
f(z, w) = (z, zw, zw^2, zw^3, w^4).
\]

\[
g(z, w) = (-w^2, zw, -zw^2, z^2 w, z^2).
\]

With $c$ denoting cosine and $s$ denoting sine, put

\[
H_t(z, w) = (cz - sw^2, zw, (cz - sw^2)(sz + cw^2), zw(sz + cw^2), (sz + cw^2)^2).
\]

When $\sin(\theta) = t = 0$ we obtain $f$ and when $t = 1$ we obtain $g$. 
Example
Put $f(z) = z$. Let $g(z)$ be the Whitney map:

$$g(z) = (z_1, z_2, \ldots, z_{n-1}, z_1 z_n, z_2 z_n, \ldots, z_n^2).$$

Then $f$ and $g$ are homotopic in dimension $2n$, but not in dimension $2n - 1$. Furthermore $f$, $g$ are spherically inequivalent.

This family arose in my work many years ago. It also arises in several results of Huang-Ji. Assume $n \geq 3$. Then there are precisely two spherical equivalence classes when $N = 2n - 1$; those equivalent to $f$ and those equivalent to $g$. Furthermore, they are able to establish results assuming only that $f$ is $C^1$ at the sphere, rather than that it is rational.
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**Theorem**

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**Sketch of Proof.**

Step 1 is a degree bound: If \(n \geq 2\), and \(f\) is a proper rational map from \(B_n\) to \(B_N\), then

\[
d \leq \frac{N(N - 1)}{2(2n - 3)}.
\]

(This bound is not sharp. Long ago I conjectured certain bounds.
Still open.)
Step 2. Let \( f = \frac{p}{q} \) be proper. Put \( R = \|p\|^2 - |q|^2 \).

\( R \) is of degree at most \( d \) in \( z \), of total degree at most \( 2d \), and is divisible by \( \|z\|^2 - 1 \). \( R \) determines a Hermitian form on the finite-dimensional vector space of polynomials of degree at most \( d \). This form has one negative eigenvalue.

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Equate Fourier coefficients in \( n \) variables after replacing \( z \) by \( e^{i\theta} z \). Obtain various linear conditions on the inner products of the vector coefficients. Putting it together gives finitely many polynomial inequalities on the coefficients.
Step 3. Conclusion: A proper rational map corresponds to the intersection of the unit sphere in a finite-dimensional real vector space with a set described by finitely many polynomial inequalities. Such a set is semi-algebraic and can have at most a finite number of components. Each component corresponds to a collection of homotopic rational proper mappings with target dimension at most \( N \).
Theorem
(D’Angelo-Lebl) Let $H_t$ be a homotopy of rational proper maps between balls. Then either all the maps are spherically equivalent or there are uncountably many spherical equivalence classes.

Proof.
Brief sketch: First step is to show that the set of $t$ in $[0, 1]$ for which $H_t$ is spherically equivalent to $H_0$ is closed. Second step is to quote a result of Sierpinski: $[0, 1]$ is not a countable union of (non-empty) disjoint closed sets.
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Sierpinski theorem: Let $X$ be a compact, connected Hausdorff space. If $\{F_n\}$ is a closed cover of pairwise disjoint subsets, then one of these sets is $X$ (and the others are empty).
Hermitian analogues of Hilbert’s 17-th problem:
Assume \( r(z, \bar{z}) \) is a polynomial and \( r > 0 \) on an algebraic set.
Does \( r(z, \bar{z}) = ||f(z)||^2 \) there?
Yes for a sphere, but not in general for strongly pseudoconvex hypersurfaces. (Putinar-Scheiderer, D’Angelo-Putinar)
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Representation theory and algebraic combinatorics:
For what finite subgroups $\Gamma$ of $\mathcal{U}(n)$ is there a non-constant proper rational maps between balls?
Group must be cyclic, and most representations are ruled out. (Lichtblau, D’Angelo-Lichtblau) But things change if we allow hyperquadrics as targets.
Variational principle. Define a functional $\Phi$ by

$$\Phi(F) = \int_\Omega \det(F_{zj \bar{z}_k}) dV.$$ 

When is

$$\Phi(F) \leq \Phi(F + \rho h)?$$

Connected with volumes of images of proper maps between balls.
Figure: Overlapping image of disk
Why consider hyperquadrics?
1) Natural generalization of the sphere.
2) According to Treves, they are a treasure trove.
3) Group invariant maps:
Let $\Gamma$ be a finite subgroup of $U(n)$. Can we find a non-constant proper rational map between balls that is $\Gamma$-invariant? In general no; in fact (Lichtblau) $\Gamma$ must be cyclic, and furthermore, $\Gamma$ must be represented in one of two ways.
BUT, if we allow the target to be a hyperquadric, then there are no restrictions on $\Gamma$, as long as we have enough eigenvalues of both signs.
Let \( \mathbb{H}_I^n \) denote the real submanifold of \( \mathbb{C}^n \) defined by

\[
\text{Re}(w) = -\sum_{j=1}^{l} |z_j|^2 + \sum_{j=l+1}^{n-1} |z_j|^2.
\]

The case \( l = 0 \) corresponds to the Heisenberg group, which is locally biholomorphically equivalent to the unit sphere. This hypersurface is locally equivalent to \( Q(a, b) \), defined by

\[
\sum_{j=1}^{a} |\zeta_j|^2 - \sum_{j=a+1}^{a+b} |\zeta_j|^2 = 1.
\]

Here \( b = l \) and \( a = N - l \).

Basic problem: Find the maps from \( \mathbb{H}_I^n \) to \( \mathbb{H}_L^N \).
Theorem (Baouendi-Huang)

Assume $n - 1 > l > 0$ and $1 < n < N$. If $f : \mathbb{H}^n_l \to \mathbb{H}^N_l$ is holomorphic and preserves sides, then $f$ is either constant or the composition of an automorphism with the standard linear embedding from $\mathbb{C}^n \to \mathbb{C}^N$. 

Baouendi, Ebenfelt, and Huang later considered the case where the number $l$ of negative eigenvalues in the domain hyperquadric and the number $l'$ in the target hyperquadric differ. Assume that this difference is small. In certain circumstances, the only possible non-constant maps are the compositions of the maps in (*) with automorphisms.
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For any holomorphic map $\psi$, the map

$$(z, w) \to (z, \psi(z, w), 0, \psi(z, w), 0, w)$$

takes $\mathbb{H}_l^n$ to $\mathbb{H}_l'^N$, simply because of cancellation.
Theorem (JPD)

For each odd positive number $2p + 1$, there is a positive integer $N(p)$ and a polynomial $g_p$ of degree $2p$ such that

$$g_p : Q(2, 2p + 1) \rightarrow Q(N(p), 2p + 1),$$

and such that $g_p$ maps to no hyperquadric $Q(a, b)$ with $a < N(p)$ or $b < 2p + 1$. 

This result does not contradict the theorem of Baouendi-Huang. This mapping $g_p$ does not preserve sides of the hyperquadric. By their result, if $g_p$ also preserved sides, it would have to be linear. In other words, the hypothesis of preserving sides is a kind of maximum principle, which automatically holds for sphere maps.
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The mapping is constructed as follows. First consider a cyclic group of order $2p$, represented as the subgroup of $U(2)$ generated by a diagonal matrix whose eigenvalues are $\omega$ and $\omega^2$, where $\omega$ is a primitive $p$-th root of unity. The invariant polynomial $\Phi_\Gamma$ has the following formula:

$$\Phi_\Gamma(z, \overline{z}) = 1 - \prod_{j=0}^{2p-1} (1 - \omega^j |z_1|^2 - \omega^{2j} |z_2|^2).$$
Putting $x = |z_1|^2$ and $y = |z_2|^2$ yields a polynomial $f(x, y)$ in two real variables $x, y$ with these properties:

- $f(x, y) = 1$ on $x + y = 1$.
- $f(\omega x, \omega^2 y) = f(x, y)$.
- $f$ has $p + 1$ positive terms and 1 negative term, namely $-y^{2p}$.
- $f(-x, y) = f(x, y)$.

In fact, $f$ has the explicit formula

$$\left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^{2p} + \left(\frac{x - \sqrt{x^2 + 4y}}{2}\right)^{2p} - y^{2p}.$$
Consider the polynomial \( F \) in \( 2p + 3 \) variables defined by

\[
F(x_1, \ldots, x_{2p+1}, y_1, y_2) = f\left(-\sum_{j=1}^{2p+1} x_j, y_1 + y_2\right).
\]

The polynomial \( F \) is 1 on the set

\[
y_1 + y_2 - \sum_{j=1}^{2p+1} x_j = 1.
\]

Replace each \( x_j \) by \(|z_j|^2\) and similarly for \( y_1 \) and \( y_2 \); this set becomes the hyperquadric \( Q(2, 2p + 1) \). The polynomial \( F \) has \( 2p + 1 \) negative terms, arising from expanding \((y_1 + y_2)^{2p}\). Hence the number of negative terms is preserved.
The polynomials mapping Lens spaces to hyperquadrics have connections with algebraic combinatorics and number theory. Consider a cyclic group $\Gamma(p, q)$ of order $p$, represented in $U(2)$ by the matrices

$$
\begin{pmatrix}
\omega & 0 \\
0 & \omega^q
\end{pmatrix}.
$$

Here $\omega^p = 1$.

Then the polynomial

$$
\Phi_{\Gamma}(z, \bar{z}) = 1 - \prod_{j=0}^{p-1} (1 - \omega^j |z_1|^2 - \omega^{qj} |z_2|^2) = f(x, y)
$$

mapping $S^3/\Gamma$ to a hyperquadric has many properties: $f(x, y)$ has integer coefficients, and $f(x, y)$ is congruent to $x^p + y^p$ modulo $p$ if and only if $p$ is prime.