COMPLEX VARIABLES IN THE CURRICULUM

JOHN P. D’ANGELO

Abstract. We discuss why and how mathematicians must revitalize the role of complex analysis in the curriculum, at several levels, in both pure and applied courses. We offer many specific examples, several of which are new, that instructors can use in their classes. We conclude with several recommendations, including revising the syllabus of Calculus II.

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1. Introduction

Courses in complex variables belong in the curriculum of students in mathematics, physics, and much of engineering for several reasons. The subject reveals mathematical art at its finest, has applications throughout the sciences, and is surprisingly accessible. Furthermore, well-taught classes in complex variables help unify apparently unrelated topics into a coherent whole. It therefore feels wrong that the subject of complex analysis is playing a diminishing role in the curriculum while it is simultaneously increasing its role in modern applications. Perhaps articles in this special issue of PRIMUS will help rectify things.

This note provides many examples one can employ in a class, assign as homework, or use as student projects. These suggestions are but a few of the many possibilities for showing the subject to be interesting, useful, modern, and elegant. The note ends with some conclusions; for example, Calculus II needs to be pruned and should include an early discussion of complex numbers.

2. The elegance of complex analysis

Example 2.1 was presented at the Santa Barbara meeting to help stimulate discussion. The reader should notice how one example provides many opportunities for unifying ideas.

Example 2.1 (Roller coaster track). Consider parametric equations for a spiral used to model a roller coaster track. We want the curvature of the track to be proportional to time, in order that the curve spirals as in Figure 1. Recall from Calculus III the formula for curvature

\[ \kappa = \frac{||v \times a||}{||v||^3} \]

in terms of velocity and acceleration. We use a complex parametric equation

\[ z(t) = \int_0^t e^{iu^2} \, du = \left( \int_0^t \cos(u^2) \, du, \int_1^t \sin(u^2) \, du \right), \quad (1) \]
because it provides simplicity and insight. One sees that the velocity $v$ and acceleration $a$ vectors are given by

$$v = z'(t) = e^{it^2} = (\cos(t^2), \sin(t^2))$$

$$a = z''(t) = 2ite^{it^2} = 2t(-\sin(t^2), \cos(t^2)).$$

It is obvious from these formulas that $||v|| = 1$ and that $v$ is orthogonal to $a$. Hence $||v \times a|| = ||a||$ and the curvature at time $t$ is given by $||a||$, which is evidently $2t$.

The reader should appreciate the simplicity of the complex variable approach here. It also highlights that multiplication by $i$ corresponds to a ninety degree rotation, a crucial fact in the distinction between direct and alternating current. This fact fits nicely into many applied complex variable courses.

![Figure 1. Roller coaster track](image)

We now wish to find the point to which the curve spirals. The complex form of the parametric equation in (1) suggests a Gaussian integral. In order to make it a Gaussian, we must replace $iu^2$ with $-\nu^2$; to do so we simply put $u = \sqrt{ir} = e^{\frac{i\pi}{4}}r$ and change variables. By Cauchy’s theorem, the line integral around the contour shown in Figure 2 is 0; it is easy to verify that the limit of the integral along the circular arc tends to 0 as $R \to \infty$. Hence the integral along the real axis is the same as the integral along the forty-five degree line. But this integral becomes a Gaussian, which we evaluate by the usual trick from Calculus III. Since $\sqrt{i} = \frac{1+i}{\sqrt{2}}$, we obtain the value

$$\frac{1+i}{\sqrt{2}} \sqrt{\frac{\pi}{2}} = (1+i) \sqrt{\frac{\pi}{8}}$$

for the limit point of the spiral.
Conformal mapping is an important topic in pure and applied complex variable texts. Color pictures add some excitement to the topic, but elegance must be maintained.

**Example 2.2.** What do analyticity and the Cauchy-Riemann equations have to do with conformality? Many texts on complex variables for applications make heavy weather over this simple point. We first recall the geometric meaning of multiplication by a complex number and remark that formula (3) should be in the toolkit of every engineer and physicist. If we use polar coordinates to write $z = |z|e^{i\theta}$ and $w = |w|e^{i\phi}$, we obtain

$$zw = |z||w|e^{i(\theta+\phi)}.$$  \hspace{1cm} (3)

Multiplication of $z$ by $w$ (assumed non-zero) therefore adds the argument of $w$ to the argument of $z$, and hence preserves angles between vectors.

What does it mean for a function $f$ to be (complex) differentiable at $z_0$? The meaning is that $f$ is approximately (complex) linear there, namely

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \text{error}.$$  

(Here, as in Calculus I, the error tends to 0 faster than $|h|$.) Hence the change in $f$ is infinitesimally given by multiplication: $h \mapsto f'(z_0)h$. Since multiplication by a non-zero complex number rotates, and hence leaves the angle between vectors unchanged, the same holds infinitesimally for a complex analytic function as long as its derivative does not vanish.

There is no need to write $u_x = v_y$ and $u_y = -v_x$ here. The chain rule is superfluous. No matrices appear. The above explanation is more fundamental than any of these things. This way of thinking provides one of many places where complex variables connect with linear algebra.

Power series form a significant part of any complex variable course. How can one make this topic interesting?

**Example 2.3** (The Z-transform). Let us first consider two ways to describe a number $x$ in $[0, 1)$. We could write $x$ as a decimal expansion

$$x = .a_1a_2a_3\ldots = \sum_{j=1}^{\infty} \frac{a_j}{10^j},$$
or we could consider the sequence \( \{a_n\} \) of its decimal digits. For a mathematician there is no difference. Similarly, given an arbitrary sequence of real or complex numbers, there should be no difference between considering the formal power series (the ordinary generating function of the sequence)
\[
\sum_{n=1}^{\infty} a_n z^n
\]
and the sequence of its coefficients. When the series converges, complex analysis provides several ways to go back and forth. One of these ways is the Cauchy integral formula for derivatives. Another way is to regard the powers of \( z \) as providing a complete orthogonal system for the square integrable (with respect to area measure) complex analytic functions on a disk; the coefficients are then inner products. Again we connect complex variables with linear algebra.

Engineers prefer the so-called \( Z \)-transform, namely the series
\[
\sum_{n=1}^{\infty} a_n z^{-n}.
\]  
(4)

Doing so makes it a bit harder to go back and forth, but has one advantage. Using the \( Z \)-transform to solve constant coefficient recurrence relations appears completely analogous to using the Laplace transform to solve constant coefficient linear ODEs. Passing between the time domain and the frequency domain is not much different from passing between the decimal digits of a number and the number itself. Yet engineering texts make a huge fuss. Complex variables courses thus provide a great opportunity to show how power series, thought of as generating functions for their coefficients, get used in applied mathematics.

In complex variable classes the author often derives Binet’s formula for the Fibonacci numbers using generating functions and relate the poles of the rational function \( \frac{1}{1-z-z^2} \) to the golden ratio. One can develop this set of ideas in many ways, ranging from how Fibonacci numbers arise in biology and art to solving the discrete difference equations in signal processing.

The Cauchy theory is a crucial part of any complex variable course. Cauchy himself appears to have thought of his integral formula in a fashion similar to how physicists regard the Dirac delta function.

**Example 2.4.** Let \( f \) be a continuous function, for example defined on the real line. How do we evaluate \( f(0) \)? A mathematician simply writes \( f(0) \). A physicist or engineer samples \( f \) near 0 and takes an average. For example, one might convolve with a square pulse. In other words,
\[
f(0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} f(t) dt. \tag{5.1}
\]

Engineers and physicists write (5.1) as
\[
f(0) = \int_{-\infty}^{\infty} \delta(t) f(t) dt. \tag{5.2}
\]

It is well-known that no such function \( \delta \) exists, yet the formula (5.2) appears throughout physics and engineering. That Cauchy anticipated this idea via his integral formula nearly 200 years ago should be mentioned in complex variable classes!
These examples and many similar ones reveal a gap between the approaches of the mathematician and the engineer. Consider for example generating functions and Laplace transforms. To a mathematician, each amounts to a vector space isomorphism: the map taking a sequence to a formal power series and the map taking a function of time to its Laplace transform, a function of frequency. Mathematicians regard the objects as essentially the same, whereas engineers regard the time domain and frequency domain as very different.

Good teaching of complex variables and linear analysis (not a typo!) can bridge this gap! Linear algebra classes fail to make the point, partly because few linear algebra courses mention infinite-dimensional settings. The linear mappings used in applied mathematics (Laplace transform, Fourier transform, $Z$-transform, convolution, etc.) typically arise in infinite dimensional settings. But these mappings appear naturally in complex variable courses. Unifying topics while teaching linear algebra seems possible and especially valuable.

One beautiful example comes from basic calculus. If $V$ is the vector space of all polynomials in one variable, $I$ denotes the identity, $D$ denotes differentiation, and $J$ denotes the definite integral, then $DJ = I$ but $JD \neq I$. This simple example reveals a fundamental difference between finite and infinite dimensions. But the similarities are striking as well. We take functions of matrices (operators) by diagonalizing them; we do the same in infinite dimensions via spectral methods. Once the author asked a class “How do we take half a derivative?”. A bright engineering student said “multiply by $\sqrt{S}$”. (In other words, diagonalize differentiation by using the Laplace transform to pass from the time domain to the frequency domain.) Such connections need to made throughout the curriculum. See [D4] for many possibilities. The teaching of linear algebra needs to be improved in order to support complex variables!

One of the best ways to proceed is to prune Calculus II. Focusing for too long on techniques of integration clutters the course. One should spend some time on linear algebra and complex numbers during Calculus II courses. Electrical engineers at Illinois have said that they want a one-semester math course that does linear algebra, one complex variable, and vector analysis! Doing so would be possible if the syllabus of Calculus II included appropriate introductions to these topics.

The author has attempted to bridge this gap via notes [D4] called “Linear and complex analysis for applications”. These notes evolved from teaching “Advanced Engineering Mathematics” in Spring 2014.

3. HOW TO REVITALIZE COMPLEX VARIABLES

At the Santa Barbara meeting, Paul Newton, an aerospace engineer and expert in non-linear fluid flow, noted and described one problem with the teaching of complex variables. The mathematics feels like 19-th century material to the students. The basics are that old. The Cauchy integral formula was proved by 1825. Cauchy, who was trained as an engineer, regarded this formula as an analogue of what is now thought of as the Dirac delta function, long before either Maxwell’s work unifying Electricity and Magnetism or Dirac’s book on quantum mechanics. Even the proof of the formula suggests sampling a function in order to evaluate it.

Most mathematicians either discuss no applications in a complex variables course, or stick to 19-th century examples. Yet there are many possibilities for invigorating the course via modern applications. Focusing on pretty computer-generated
pictures without meaningful applications is not the answer. While every student should realize that squaring maps the first quadrant to the upper half-plane, finding, for example, the image of random sets in $\mathbb{C}$ under squaring is a waste of effort, and may in fact impede intellectual development. Not all pictures are equally useful. Asking students to have the computer generate pictures of arbitrary images amounts to saying “stop thinking”.

Complex analysis offers an opportunity to make elegant, important insights about both pure and applied mathematics. The teacher should behave as a CEO for a company, wanting to make a good impression on potential users of the product.

Discussion of the following topics should mitigate the feeling of irrelevance and also reveal complex analysis to be of current interest. The author has thrice taught a course for honors freshmen called “Complex geometry”. The book [D3] developed from these courses. Each student must write a paper and present it to the class; several of the following topics have been used for these projects. Some of the other topics were developed in the Advanced Engineering Mathematics course which led to [D4].

- Fractal dimensions are used in crop sciences; one can regard the root system of a crop as a self-similar fractal. Knowing its fractal dimension enables crop scientists to determine where to water or provide nutrients.
- When discussing linear fractional transformations, one could introduce the $ABCD$-matrices of transmission lines or the ray-transfer matrices of optics. One can analyze complicated optical devices by regarding each component as a two-by-two matrix and regarding the device as the matrix product. Here the matrix product corresponds to the composition of linear fractional transformations.
- Quaternions are used in computer graphics. At some point in a good complex variable class, the instructor could introduce the quaternions and mention how they arise in physics and computer graphics.
- The connection between the exponential function and trig dominates much of applied mathematics. This topic arises throughout any good complex variable class, and its usefulness should be mentioned on multiple occasions. The connection is greatly enhanced when one revisits techniques of integration. See Example 4.1. Also, many students have wondered why $\int \frac{dt}{t^2+1}$ is an inverse tangent while $\int \frac{dt}{t^2-1}$ involves logs. Faculty should let the students discover this connection throughout the course, reconnecting with complex exponentials and Fourier series as much as possible. When discussing residues, one should pause to define the Laplace and Fourier transforms and compute some examples using residues. In particular, one can help revitalize the complex variable curriculum simply by showing how these techniques are used in physics and engineering.
- One could discuss the Lambert $W$-function; here $W$ is a branch of the solution to $z = we^w$. This simple example of an non-elementary function has unimaginably diverse applications. It arises in combinatorics and computer science in enumeration problems, in number theory, in physics such as in the study of the Planck, Bose-Einstein, and Fermi-Dirac distributions, in delay differential equations, and in biochemistry such as enzyme kinetics.
Its ubiquity makes one wonder whether it should be regarded as an elementary function! In any event, it seems worth at least mentioning the possible applications. See [CGHJK].

- Any good class in complex analysis should mention the Dirichlet problem, and should make the point that the Riemann mapping theorem enables one to solve this basic PDE for many domains by reducing to the unit disk. There the solution naturally introduces Fourier series and the Poisson kernel. One sees again how approximate identities make the Dirac delta function into a precise mathematical object.
- Fourier series and the Fourier transform have been popular topics for student projects. For example, have the student learn summation by parts and use it to investigate the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$.
- Derive formulas for $\sum_{n=1}^{N} n^p$ using Bernoulli numbers.
- Investigate the prime number theorem.
- The following might work for a project. Watch the you tube video where physicists “prove” that

$$1 + 2 + 3 + 4 + \ldots = \frac{-1}{12}.$$  

Then provide a clear explanation of what is going on. Hint: for positive $s$,

$$\sum_{n=0}^{\infty} ne^{-ns} = \frac{e^s}{(1 - e^s)^2} = \frac{1}{s^2} + \frac{-1}{12} + \ldots$$

where the dots denote an analytic function.

We mathematicians can improve our teaching simply by becoming more aware of how other scientists use our ideas. As suggested above, the Cauchy integral formula can be thought of as evaluating a function by sampling it and averaging the results. The Dirac delta function gives one an opportunity to discuss the interchange of limiting operations. All students should know that

$$1 = \lim_{n \to \infty} \lim_{m \to \infty} \left(\frac{1}{n}\right)^{\frac{1}{m}} \neq \lim_{m \to \infty} \lim_{n \to \infty} \left(\frac{1}{n}\right)^{\frac{1}{m}} = 0. \quad (6)$$

The author recalls an amusing conversation with professors in physics and psychology. The psychologist mentioned discussing the meaning of $0^0$ in a psych class. The author noted formula (6) to try to explain the subtlety behind interchanging limits. The physicist had just published a paper where the physics depended upon which of two small parameters went to zero first! Students often use rather sophisticated mathematical concepts in physics and engineering classes, but their understanding would be enhanced if we mathematicians taught these topics also. We fail by postponing interesting topics and by omitting applications when we introduce these topics.

4. Pedagogical issues

It is time to consider pedagogy. The main problem with mathematics teaching goes well beyond complex variables. High school math does not provide the right background, and the calculus sequence includes many useless topics. The best way to revitalize complex variables is to introduce them in the calculus sequence and then to apply them throughout the mathematics curriculum.
Another crucial point concerns linear algebra, which represents a missed opportunity. Here is one way to think about it. Students spend several years in high school learning about the equations of lines in the plane. It is then somewhat sensible to approximate curves by their tangent lines. By contrast, Calculus III courses spend less than two weeks on straight objects in Euclidean space, but then expect students to catch on to linear approximation. One way to improve things is to get complex variables into the curriculum sooner. We can regard the complex number $z$ in several ways; one of these is as the 2-by-2 matrix \[
\begin{pmatrix}
x & -y \\
y & x
\end{pmatrix}.
\] Identifying a point in the plane with a single letter helps prepare for abstract thinking. The connection between exponentiation and trig also makes a positive impression on many students. We must exploit this beautiful and powerful idea.

**Example 4.1.** Calculus books waste considerable effort on integrals involving the trig functions. Who cares what the indefinite integral of $\cos^{2N}$ is? The definite integral (7), however, matters:
\[
\int_0^{2\pi} \cos(\theta)^{2N} \, d\theta = \frac{2\pi}{2^{2N}} \binom{2N}{N}.
\] (7)

The formula $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$ enables students (assuming they recall the binomial theorem) to compute (7) in their heads, and streamlines the course.

Many ideas arising in applied math have elementary roots, yet both pure and applied courses often miss these connections. Mathematicians need to know how our subject is used elsewhere and take advantage!

**Example 4.2.** Convolution plays a crucial role in physics and engineering, especially when studying approximate identities or linear time-invariant systems (LTI systems). When teaching convolution one should remind students how to multiply polynomials (or power series). One can do grade school arithmetic in this fashion! The reader might try to compute $493 \times 124$ in his head as follows:
\[
493 \times 124 = (4x^2 + 9x + 3)(x^2 + 2x + 4) = 4x^4 + 17x^3 + 37x^2 + 42x + 12 \bigg|_{x=10} = 61132.
\]

5. **Conclusion**

Complex analysis is a beautiful and applicable subject. Perhaps its teaching has become a bit stagnant. It seems easily possible to revitalize it by providing more modern applications and by encouraging student projects using complex analysis, along the lines of the examples suggested here. Only a minimal amount of technology is needed. A crucial piece of the revitalization involves the teaching of other subjects! Despite countless recommendations on the teaching of calculus and linear algebra, pedagogy in those areas remains less than optimal. One should prune the calculus sequence, introduce both complex numbers and linear algebra earlier in the curriculum, and revise the linear algebra syllabus so as to create a more unified approach to undergraduate mathematics teaching. Perhaps the most important recommendation involves attitude. Mathematicians should show off the applications of our subject while maintaining its intellectual rigor. Complex variable theory offers a striking opportunity to do both simultaneously.
6. REFERENCES

The following references provide worthwhile discussion of many topics in complex analysis and linear algebra. Perhaps [A] and [SS] are two of the best complex analysis books ever written. Books such as [D3], [D4], [E], [F], [G], [GS], [KM], [HM] link pure and applied math. Of the profusion of linear algebra books, [HK] and [St] standout, for different reasons. Books such as [D1], [D2] and [K] provide accessible accounts of elementary complex analysis from unusual perspectives. The article [CGHJK] offers a long diverse list of applications of the Lambert W-function.


Dept. of Mathematics, Univ. of Illinois, 1409 W. Green St., Urbana IL, 61801, USA
E-mail address: jpdbmath.uiuc.edu