When is the sum of inverses the inverse of the sum?

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It is October 26, 2012, Martin Gardner day at Univ. of Illinois.
Here is something he would have enjoyed.
Teachers sometimes see $\frac{1}{x} + \frac{1}{y} = \frac{1}{x+y}$ on exam papers.
Egad! There are no real numbers satisfying this identity.

There are complex numbers that do.
For example, let $\omega = \frac{-1+i\sqrt{3}}{2}$. Then $\frac{1}{\omega} = \overline{\omega}$.
Hence ω is on the circle.
Also $1 + \omega + \omega^2 = 0$. Draw a picture!
Then $\frac{1}{1} + \frac{1}{\omega} = \frac{1}{1+\omega}$ because

$$(1 + \omega)(1 + \frac{1}{\omega}) = 1 + \omega + \overline{\omega} + \omega\overline{\omega} = 1 + \omega + \omega^2 + 1 = 1.$$

Can we can find linear operators (matrices) which satisfy

$$A^{-1} + B^{-1} = (A + B)^{-1}? \qquad (*)$$

Definition

A real vector space V *admits a complex structure* if there is a linear map $J : V \to V$ such that $J^2 = -I$.

A finite-dimensional real vector space admits a complex structure if and only if its dimension is even. For example, the linear transformation $J : \mathbf{R}^2 \to \mathbf{R}^2$ corresponding to the complex structure is given by the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Theorem

Let V be a vector space over $\mathbf{R}$. Then there are invertible linear transformations A, B on V satisfying

$$(A+B)^{-1} = A^{-1} + B^{-1} \qquad (*)$$

if and only if V admits a complex structure.

Proof: Invertible A, B satisfying (*) exist if and only if

$$I = (A + B)(A^{-1} + B^{-1}) = I + BA^{-1} + I + AB^{-1}.$$

Put $C = BA^{-1}$. The condition (*) is therefore equivalent to finding C such that $0 = I + C + C^{-1}$, which is more easily expressed as $0 = I + C + C^2$. Notice that C behaves like a complex cube root of unity.

Suppose such C exists. Put $J = \frac{1}{\sqrt{3}}(I + 2C)$. Then we have

$$J^2 = \frac{1}{3}(I + 2C)^2 = \frac{1}{3}(I + 4C + 4C^2) = \frac{1}{3}(I - 4I) = -I.$$

Hence V admits a complex structure.
Conversely, if V admits a complex structure, then J exists with $J^2 = -I$. Put $C = \frac{-I+\sqrt{3}J}{2}$; then $I + C + C^2 = 0$.

Corollary

There are n by n matrices satisfying $A^{-1} + B^{-1} = (A + B)^{-1}$ if and only if n is even!

Variations:

The theorem fails over the rationals, because $\sqrt{3}$ is irrational.

It fails over fields of characteristic 2 and 3 as well.

Interested readers could also analyze

$$\sum_{j=1}^{n}(A_j)^{-1} = (\sum_{j=1}^{n} A_j)^{-1}.$$

We give another variant, important in physics, to close.

There are no real numbers x and y with $xy - yx = 1$.
There are no finite-dimensional matrices A, B with
$[A, B] = AB - BA = I$. (Take the trace!)
Yet, finding linear operators with this property is crucial to quantum mechanics.
Fact: No *bounded* operators satisfy $[A, B] = AB - BA = I$.
Here is an example that does:
$A = \frac{d}{dx}$ and $B = x$ on the space of differentiable functions.
Then $(AB - BA)(f) = (xf)' - xf' = f$ and hence $[A, B] = I$.
Remark: $AB - BA$ is not a Swedish pop group but $ABBA$ is.

Perhaps Gardner would enjoy the following conclusion:
When a math problem has no solutions, don't give up. Instead, generalize until it does have solutions.