416 HOMEWORK: D’ANGELO

1. Consider \( \mathbb{R}^4 \). Show that the set \( W_1 \) of vectors of the form \((x, -x, y, z)\) forms a subspace. Also show that the set \( W_2 \) of vectors of the form \((a, b, -a, c)\) is a subspace.
   a. Find bases for \( W_1 \) and \( W_2 \).
   b. Find bases for \( W_1 + W_2 \) and \( W_1 \cap W_2 \).
   c. Verify the formula
      \[
      \dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).
      \]

2. Let \( V \) be the vector space of functions \( f : \mathbb{R} \to \mathbb{R} \). Verify (as we did the first day of class) that \( V \) can be decomposed
   \[
   V = V_e \oplus V_o
   \]
in terms of even and odd functions. Note that you must show that the intersection of the two space is 0 alone, in order to use the \( \oplus \) notation.

3. Let \( V \) be a vector space over the field \( \mathbb{F} \). Let \( L : V \to V \) be a linear transformation. For \( \lambda \in \mathbb{F} \), let \( E_\lambda \) denote the set of vectors \( v \) for which \( Lv = \lambda v \). The subspace \( E_\lambda \) is called the eigenspace corresponding to \( \lambda \) and \( \lambda \) is called an eigenvalue of \( L \).
   a. Verify that \( E_\lambda \) is a subspace.
   b. Suppose that \( V = \mathbb{R}^2 \) and that \( L \) is given by the matrix
      \[
      \begin{pmatrix}
      -1 & 8 \\
      -4 & 11
      \end{pmatrix}
      \]
      (with respect to the usual bases). Show that 3 and 7 are eigenvalues for \( L \).
   c. Find a basis for each of the \( E_\lambda \) when \( L \) is as above.

4. Let \( a, b \) be real numbers. Consider the matrix
   \[
   \begin{pmatrix}
   a & -b \\
   b & a
   \end{pmatrix}
   \]
   Show that the set of these matrices (under the usual sum and product) forms a field. What is the additive identity \( 0 \)? What is the multiplicative identity \( 1 \)? Find a matrix \( J \) of this type with \( J^2 = -1 \). Determine the eigenvalues of \( J \). (They are complex rather than real.)

5. Suppose \( L : V \to V \) is linear. Consider the following two properties:
   a) \( L(Lv) = 0 \) implies \( Lv = 0 \)
   b) \( \mathcal{N}(L) \cap \mathcal{R}(L) = 0 \)
   Prove that a) holds if and only if b) holds.
   Give an example of such an \( L \) for which both \( \mathcal{N}(L) \) and \( \mathcal{R}(L) \) are positive dimensional.

6. For \( n \geq 2 \), give an example of an \( n \)-by-\( n \) matrix \( A \) such that \( A^n = 0 \) but \( A^{n-1} \neq 0 \).