TOPOLOGICAL FUKAYA CATEGORY AND MIRROR SYMMETRY FOR PUNCTURED SURFACES

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Abstract. In this paper we establish a version of homological mirror symmetry for punctured Riemann surfaces. Following a proposal of Kontsevich we model A-branes on a punctured surface Σ via the topological Fukaya category. We prove that the topological Fukaya category of Σ is equivalent to the category of matrix factorizations of the mirror LG model (X, W). Along the way we establish new gluing results for the topological Fukaya category of punctured surfaces which might be of independent interest.

Contents

1. Introduction

The Fukaya category is an intricate invariant of symplectic manifolds. One of the many subtleties of the theory is that pseudo-holomorphic discs, which control compositions of morphisms in the Fukaya category, are global in nature. As a consequence, there is no way to calculate the Fukaya category of a general symplectic manifold by breaking it down into local pieces.[1] In the case of exact symplectic manifolds, however, the Fukaya category is expected to have good local-to-global properties. If $S = T^*M$ is the cotangent bundle of an analytic variety this follows from work of Nadler and Zaslow. They prove in [NZ, N] that the (infinitesimal) Fukaya category of S is equivalent to the category of constructible sheaves on the base manifold M. This implies in particular that the Fukaya category of S localizes as a sheaf of categories over M.

Recently Kontsevich [K] has proposed that the Fukaya category of a Stein manifold S can be described in terms of a (co)sheaf of categories on a skeleton of S. A skeleton is, roughly, a half-dimensional CW complex X embedded in S as Lagrangian deformation retract. According to Kontsevich X should carry a cosheaf of categories, which we will denote $\mathcal{F}^{top}$, that encodes in a universal way the local geometry of the singularities of X. He conjectures that the global sections of $\mathcal{F}^{top}$ on X should be equivalent to the wrapped Fukaya category of S.

Giving a rigorous definition of the cosheaf $\mathcal{F}^{top}$ is subtle. Work of several authors has clarified the case of punctured Riemann surfaces [DK, N1, STZ], while generalizations to higher dimensions have been pursued in [N4, N5]. The theory is considerably easier in complex dimension one because skeleta of punctured Riemann surfaces, aka ribbon graphs or spines, have a simple and well studied combinatorics and geometry, while the higher dimensional picture is only beginning to emerge [RSTZ, N2]. Implementing Kontsevich's ideas, the formalism developed in [DK, N1, STZ] defines a covariant functor $\mathcal{F}^{top}(-)$ from a category of ribbon graphs and open inclusions to triangulated dg categories.

An important feature of the theory is that, if X and X' are distinct compact skeleta of a punctured surface Σ, there is an equivalence

$$\mathcal{F}^{top}(X) \simeq \mathcal{F}^{top}(X').$$

We will refer to $\mathcal{F}^{top}(X)$ as the *topological Fukaya category* of Σ, and we denote it $Fuk^{top}(\Sigma)$. In this paper we take $Fuk^{top}(\Sigma)$ as a model for the category of A-branes on Σ. We prove homological mirror symmetry for punctured Riemann surfaces by showing that $Fuk^{top}(\Sigma)$ is equivalent to the category of B-branes on the mirror. As a byproduct of our main result, we show that $Fuk^{top}(\Sigma)$ is equivalent to the wrapped Fukaya category in a large number of examples, giving further evidence in support of Kontsevich's conjecture.

1.1. **Hori-Vafa homological mirror symmetry.**

Let us review the setting of Hori-Vafa mirror symmetry for LG models [HV, GKR]. Let X be a toric threefold with trivial canonical bundle. The fan of X can be realized as a smooth subdivision of the cone over a two-dimensional lattice polytope, see Section 3.1.1 for more details. The height function on the fan of X gives rise to a regular map

$$W : X \to \mathbb{A}^1,$$

which is called the *superpotential*. The category of B-branes for the LG model (X, W) is the $\mathbb{Z}_2$-graded category of matrix factorizations $MF(X, W)$. The mirror of the LG-model

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[1] See however recent proposals of Tamarkin [Ta] and Tsygan [Ts].

(X, W) is a smooth algebraic curve Σ_W in $\mathbb{C}^* \times \mathbb{C}^*$, called the *mirror curve.* The following is our main result.

Theorem 1.1 (Hori-Vafa homological mirror symmetry)**.** *There is an equivalence*

$$Fuk^{top}(\Sigma_W) \simeq MF(X, W).$$

Theorem 1.1 provides a complete proof of homological mirror symmetry for punctured surfaces, provided that we model the category of A-branes via the topological Fukaya category. This extends to all genera earlier results for curves of genus zero and one which were obtained in [STZ] and [DK]. We also mention work of Nadler, who studies both directions of Hori-Vafa mirror symmetry for higher dimensional pairs of pants [N3, N4, N5].

We learnt the statement of Hori-Vafa homological mirror symmetry for punctured surfaces from the inspiring paper [AAEKO]. In [AAEKO] the authors prove homological mirror symmetry for punctured spheres. Their main theorem is parallel to our own (in genus zero) with the important difference that they work with the wrapped Fukaya category, rather than with its topological model. See also related work of Bocklandt [B]. Mirror symmetry for higher-dimensional pairs of pants was studied by Sheridan in [Sh].

Denote $Fuk^{wr}(\Sigma)$ the wrapped Fukaya category of a punctured surface Σ. Our main result combined with the main result of [AAEKO] gives equivalences

$$Fuk^{top}(\Sigma_W) \simeq MF(X, W) \simeq Fuk^{wr}(\Sigma_W),$$

for all Riemann surfaces Σ_W which can be realized as unramified cyclic covers of punctured spheres. Thus, for this class of examples, the topological Fukaya category captures the wrapped Fukaya category, corroborating Kontsevich's proposal.

Remark 1.2. When we were close to completing the project we learnt that Lee, in her unpublished thesis [Le], extends the results of [AAEKO] to all genera. Although our techniques are very different, conceptually the approach pursued in this work and in Lee's are closely related. The results of this paper are logically independent of those of [Le], since we use the topological version of the Fukaya category instead of the version defined in terms of pseudo-holomorphic curves. In fact, combining the results of this paper with those of [Le] implies that the topological and the wrapped Fukaya categories coincide for all punctured surfaces.

1.2. **The topological Fukaya category and pants decompositions.**

The technical core of the paper is a study of the way in which the topological Fukaya category interacts with pants decompositions. By construction $\mathcal{F}^{top}(-)$ is a cosheaf of categories on the spine of a punctured surface. So locality is built in in the definition of the topological Fukaya category. From a geometric perspective, this locality corresponds to cutting up the surface into flat polygons having their vertices at the punctures.

In this paper we prove that the topological Fukaya category of a punctured surface satisfies also a different kind of local-to-global behavior: it can be glued together from the Fukaya categories of the pairs of pants making up a pants decomposition of it. We believe that this result is of independent interest. We expect this to be a feature of the topological Fukaya category in all dimensions, and we will return to this in future work. Based on recent parallel advances in the theory of the wrapped Fukaya category [Le], this seems to be a promising avenue to compare the wrapped and the topological pictures of the category of A-branes on Stein manifolds. In order to explain the gluing formula for pants decompositions we need to sketch first a construction that attaches to a tropical curve G a category $\mathcal{B}(G)$, full details can be found in Section 3.1.3.

Let κ be the ground field. We denote $MF^{\infty}(X, f)$ and $Fuk^{top}_{\infty}(\Sigma)$ the Ind completions of the categories $MF(X, f)$ and $Fuk^{top}(\Sigma)$. We attach to a vertex v of G the category

$$\mathcal{B}(v) := MF^{\infty}(\mathbb{A}^3_{\kappa}, xyz),$$

and to an edge e of G the category

$$\mathcal{B}(e) := MF^{\infty}(\mathbb{G}_m \times \mathbb{A}^2_{\kappa}, yz),$$

where y and z are coordinates on $\mathbb{A}^2_{\kappa}$. If a vertex v is incident to an edge e there is a restriction functor $\mathcal{B}(v) \to \mathcal{B}(e)$. We define $\mathcal{B}(G)$ as the (homotopy) limit of these restriction functors.

Theorem 1.3. *Let Σ be an algebraic curve in $\mathbb{C}^* \times \mathbb{C}^*$ and let G be its tropicalization. Then there is an equivalence*

$$Fuk^{top}_{\infty}(\Sigma) \simeq \mathcal{B}(G). \tag{1}$$

As proved in [P], $MF^{\infty}(X, f)$ is a sheaf of categories for the étale topology on X. This gives rise to an expression for $MF^{\infty}(X, f)$ which is exactly parallel to (1). Our main theorem follows easily from here.

1.3. The topological Fukaya category and closed covers.

The proof of Theorem 1.3 hinges on the key observation that the cosheaf $\mathcal{F}^{top}(-)$ behaves like a sheaf with respect to a certain type of *closed covers*. This somewhat surprising property of $\mathcal{F}^{top}(-)$ is very natural from the viewpoint of mirror symmetry, because it is mirror to Zariski descent of quasi-coherent sheaves and matrix factorizations. Denote $\mathcal{F}^{top}_{\infty}(-)$ the Ind-completion of $\mathcal{F}^{top}(-)$.

Theorem 1.4. *Let X be a ribbon graph.*

- *If Z is a closed subgraph of X there are restriction functors*
$$R : \mathcal{F}^{top}(X) \to \mathcal{F}^{top}(Z), \quad R_{\infty} : \mathcal{F}^{top}_{\infty}(X) \to \mathcal{F}^{top}_{\infty}(Z).$$
- *Let Z_1 and Z_2 be closed subgraphs of X, such that $Z_1 \cup Z_2 = Z$. Assume that the underlying topological space of the intersection $Z_{12} = Z_1 \cap Z_2$ is a disjoint union of copies of S^1. Then the diagram*
$$\begin{tikzcd} \mathcal{F}^{top}_{\infty}(X) \arrow[r] \arrow[d] & \mathcal{F}^{top}_{\infty}(Z_1) \arrow[d] \\ \mathcal{F}^{top}_{\infty}(Z_2) \arrow[r] & \mathcal{F}^{top}_{\infty}(Z_{12}) \end{tikzcd}$$
is a homotopy fiber product of dg categories.

Restrictions to closed subgraphs for the topological Fukaya category have also been been considered by Dyckerhoff in [D]. From the perspective of the wrapped Fukaya category, they are closely related to the stop removal functors appearing in recent work of Sylvan [Sy]. Theorem 1.4 is a key ingredient in the proof of Theorem 1.3, which also depends on a careful study of the geometry of skeleta under pants attachments. Indeed our proof of Theorem 1.3 hinges on a recursion where, at each step, we are simultaneously gluing in a pair of pants and deforming the skeleton on the surface to make it compatible with this gluing. The topological analysis required for the argument is carried out in Section 7.

1.4. **The structure of the paper.** In section 2 we fix notations on dg categories and ribbon graphs. In Section 3 we explain the set-up of Hori-Vafa mirror symmetry and prove a key decomposition of the category of matrix factorizations, which is a simple consequence of its sheaf properties. Section 4 contains a summary of the theory of the topological Fukaya category based mostly on [DK], while in Section 5 we study restrictions functors to open and closed subgraphs, and their compatibilities. In Section 6 we prove that the topological Fukaya category can be glued from a special kind of covers by closed sub-graphs. Section 7 is devoted to a careful examination of the interactions between ribbon graphs and pants decompositions. This play a key role in the proof of the main theorem, which is contained in Section 8.

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2. Notations and conventions

We fix throughout a ground field κ of characteristic 0.

2.1. **Categories.** We refer to Section 1 of [DK] and Section 2 of [D] for a detailed summary of the homotopy theory of $\mathbb{Z}_2$-graded dg categories. We mostly follow the conventions of [DK] and [D]. The Morita theory of $\mathbb{Z}$-graded dg categories, on which the $\mathbb{Z}_2$-graded theory is closely patterned, is due to Toën [To]. The reference for ∞-categories is [Lu].

Definition 2.1. We denote $dgCat^{(2)}$ the symmetric monoidal ∞-category of κ-linear $\mathbb{Z}_2$-graded dg categories localized at Morita equivalences. If A and B are in $dgCat^{(2)}$ we denote $Fun(A, B) \in dgCat^{(2)}$ their internal Hom.

The category $dgCat^{(2)}$ is equivalent to its full ∞-subcategory of stable and idempotent complete $\mathbb{Z}_2$-graded dg categories.

Definition 2.2. We denote $dgCat^{(2),L}$ and $dgCat^{(2),R}$ the wide ∞-subcategories of $dgCat^{(2)}$ having respectively left and right adjoints as morphisms. There is an equivalence

$$(dgCat^{(2),L})^{op} \simeq dgCat^{(2),R}.$$

Definition 2.3. We denote:

- $dg\mathcal{P}r^{(2),L}$ the symmetric monoidal ∞-category of κ-linear presentable $\mathbb{Z}_2$-graded dg categories and left adjoint functors between them. If A and B are in $dg\mathcal{P}r^{(2),L}$ we denote $Fun(A, B) \in dg\mathcal{P}r^{(2),L}$ their internal Hom.

- $dg\mathcal{P}r^{(2),R}$ the symmetric monoidal ∞-category of presentable $\mathbb{Z}_2$-graded dg categories and right adjoint functors between them. As before, we denote $Fun(A, B)$ the internal Hom of two objects A and B in $dg\mathcal{P}r^{(2),R}$.
- $\widehat{dgCat}^{(2)}$ the ∞-category of (non-necessarily small) $\mathbb{Z}_2$-graded dg-categories.

Remark 2.4. The categories $dg\mathcal{P}r^{(2),L}$ and $dg\mathcal{P}r^{(2),R}$ are complete and cocomplete, and there is an equivalence: $(dg\mathcal{P}r^{(2),L})^{op} \simeq dg\mathcal{P}r^{(2),R}$. There are forgetful functors

$$dg\mathcal{P}r^{(2),L} \xrightarrow{i_L} \widehat{dgCat^{(2)}}, \quad dg\mathcal{P}r^{(2),R} \xrightarrow{i_R} \widehat{dgCat^{(2)}}.$$

Definition 2.5. We denote Ind the Ind-completion functor

$$Ind : dgCat^{(2)} \to dg\mathcal{P}r^{(2),L}.$$

If X is a scheme or a DM stack we denote

$$\mathcal{P}\mathrm{erf}^{(2)}(X) \in dgCat^{(2)}, \quad \mathcal{Q}Coh^{(2)}(X) \simeq Ind(\mathcal{P}\mathrm{erf}^{(2)}(X)) \in \mathcal{P}r^{(2),L}$$

the $\mathbb{Z}_2$-foldings of the dg categories of perfect complexes and of quasi-coherent sheaves on X.

2.2. **Ribbon graphs.** For a survey of the theory ribbon graphs see [MP], and Section 3.3 of [DK]. We will just review some standard terminology. A graph X is a pair (V, H) of finite sets equipped with the following extra data:

- An involution $\sigma : H \to H$
- A map $I : H \to V$

We call V the set of vertices, and H the set of half-edges. Let v be a vertex. We say that the half-edges in $I^{-1}(v)$ are incident to v. The cardinality of $I^{-1}(v)$ is called the valency of the vertex v. The edges of X are the equivalence classes of half-edges under the action of σ. We denote E the set of edges of σ. The set of external edges of X is the subset $E^o \subset E$ of equivalence classes of cardinality one, which correspond to the fixed points of σ. The internal edges of X are the elements of $E - E^o$. Subdividing an edge e of X means adding to X a two-valent vertex lying on e. More formally, let e be equal to $\{h_1, h_2\} \subset H$. We add a new vertex v_e to V, and two new half-edges h_1' and h_2' to H. We modify the maps σ and I by setting

$$\sigma(h_1) = h_1', \quad \sigma(h_2) = h_2', \quad I(h_1') = I(h_2') = v_e.$$

It is often useful to view a graph as a topological space. This is done by modeling the external and the internal edges of G, respectively, as semiclosed and closed intervals, and gluing them according to the incidence relations. We refer to this topological model as the underlying topological space of X. When talking about the embedding of a graph X into a topological space, we always mean the embedding of its underlying topological space.

Definition 2.6. A *ribbon graph* is a graph $X = (V, H)$ together with the datum of a cyclic ordering of the set $I^{-1}(v)$, for all vertices v of X.

If a graph X is embedded in an oriented surface it acquires a canonical ribbon graph structure. Conversely, it is possible to attach to any ribbon graph X a non-compact oriented surface inside which X is embedded as a strong deformation retract. See [MP] for additional details on these constructions. If Σ is a Riemann surface, a *skeleton* of Σ is a ribbon graph X together with an embedding $X \to \Sigma$ as a strong deformation retract.

3. Hori-Vafa mirror symmetry

In this section we review the setting of mirror symmetry for toric Calabi-Yau LG models in dimension three. Mirror symmetry for LG models was first proposed by Hori and Vafa [HV], and is the subject of a vast literature in string theory. In this paper we compare the category of B-branes on toric Calabi-Yau LG models and the category of A-branes on the mirror.

3.1. **B-branes.**

3.1.1. *Toric Calabi-Yau threefolds.* Let $\widetilde{N}$ be a $n-1$-dimensional lattice, and let P a lattice polytope in $\widetilde{N}_{\mathbb{R}} = \widetilde{N} \otimes \mathbb{R}$. Set $N := \widetilde{N} \otimes \mathbb{Z}$, and $N_{\mathbb{R}} := N \otimes \mathbb{R}$. Denote $C(P) \subset N_{\mathbb{R}}$ the cone over the polytope P placed at height one in $N_{\mathbb{R}}$. More formally, consider

$$\{1\} \times P \subset N_{\mathbb{R}} \cong \mathbb{R} \oplus \widetilde{N}_{\mathbb{R}},$$

and let $C(P)$ be the convex hull of the union of the origin $0_{N_{\mathbb{R}}}$ and $\{1\} \times P$ inside $N_{\mathbb{R}}$. Let $F(P)$ be the fan of faces of $C(P)$. The affine toric variety X_P corresponding to $F(P)$ has an isolated Gorenstein singularity. The toric resolutions of X_P are in bijection with smooth subdivisions of the cone $C(P)$. We will be interested in toric *crepant* resolutions, that is, resolutions with trivial canonical bundle.

Toric crepant resolutions of X_P are given by unimodular triangulations of P, i.e. triangulations of P by elementary lattice simplices. Any such triangulation $\mathcal{T}$ gives rise a smooth subdivision of the cone $C(P)$, which we denote $C(\mathcal{T})$: $C(\mathcal{T})$ is the set of cones on the simplices $T \in \mathcal{T}$ placed at height one in $N_{\mathbb{R}}$. Let $F(\mathcal{T})$ be the corresponding fan, and let $X_{\mathcal{T}}$ be the toric variety with fan $F(\mathcal{T})$. The variety $X_{\mathcal{T}}$ is smooth and Calabi-Yau. All toric crepant resolutions of X_P are isomorphic to $X_{\mathcal{T}}$ for some unimodular triangulation $\mathcal{T}$ of P. The following definition will be useful later on, see for instance [BJMS] for additional details on this construction.

Definition 3.1. We denote $G_{\mathcal{T}}$ the tropical curve dual to the triangulation $\mathcal{T}$ of P.

Let $\widetilde{M} = Hom(\widetilde{N}, \mathbb{Z})$ and $M = Hom(N, \mathbb{Z})$. The height function on N is by definition the projection

$$N = \mathbb{Z} \times \widetilde{N} \to \mathbb{Z}.$$

The height function corresponds to the lattice point $(1, 0) \in M \cong \mathbb{Z} \times \widetilde{M}$, which determines a monomial function

$$W_{\mathcal{T}} : X_{\mathcal{T}} \to \mathbb{A}^1_{\kappa}.$$

The category of B-branes on the Landau-Ginzburg model $(X_{\mathcal{T}}, W_{\mathcal{T}})$ is the category of matrix factorizations for $W_{\mathcal{T}}$, $MF(W_{\mathcal{T}})$. We review the theory of matrix factorizations next.

3.1.2. *Matrix factorizations.* Let X be a scheme or a smooth DM-stack and let $f : X \to \mathbb{A}^1_\kappa$ be a regular function. The category of matrix factorizations for the pair (X, f) was defined in [LP] and [O2], extending the theory of matrix factorizations for algebras that goes back to classical work of Eisenbud [E]. These references make various assumptions on f and X, which are always satisfied in the cases we are interested in. In the following X will always be smooth of finite type, and f will be flat. We will work with a dg enhancement of the category of matrix factorizations, which has been studied for instance in [LS] and [P]. We refer to these papers for additional details. We denote $MF(X, f)$ the $\mathbb{Z}_2$-periodic dg-category of matrix factorizations of the pair (X, f). It will often be useful to work with Ind-completed categories of matrix factorization.

Definition 3.2. We denote $MF^\infty(X, f)$ the Ind-completion of $MF(X, f)$,

$$MF^\infty(X, f) = Ind(MF(X, f)) \in dg\mathcal{P}r^{(2),L}.$$

The category $MF^\infty(X, f)$ has the following important descent property.

Proposition 3.3 ([P] Proposition A.3.1)**.** *Let $f : X \to \mathbb{A}^1$ be a morphism. Then the assignment*

$$U \mapsto MF^\infty(U, f|_U)$$

determines a sheaf for the étale topology.

Using Proposition 3.3 we can give give a very concrete description of $MF^\infty(X_\mathcal{T}, W_\mathcal{T})$, where $X_\mathcal{T}$ and $W_\mathcal{T}$ are as in section 3.1.1. In order to do so, we need to explain how to attach a matrix factorizations-type category to a certain class of planar graphs. This will require setting up some notations and preliminaries.

Let I be a set of cardinality three, say $I = \{a, b, c\}$. Denote

$$X_I = Spec(\kappa[t_i, i \in I]) = Spec(\kappa[t_a, t_b, t_c]),$$

and let f be the regular function

$$f = \times_{i \in I} t_i = t_a t_b t_c : X \longrightarrow \mathbb{A}^1_\kappa.$$

For all $j \in I$, let I_j be the subset $I - \{j\} \subset I$. Let U_j be the open subscheme $X - \{t_j = 0\}$, and let ι_j be the inclusion $U_j \subset X$. Denote

$$\iota_j^* : MF^\infty(X_I, f) \longrightarrow MF^\infty(U_j, f|_{U_j})$$

the restriction functor. Let f_j be the regular function

$$f_j = \times_{i \in I_j} t_i : U_j \longrightarrow \mathbb{A}^1_\kappa.$$

Note that $f|_{U_j}$ is given by $t_j f_j$.

Proposition 3.4. *There are equivalences in $dg\mathcal{P}r^{(2),L}$*

$$MF^\infty(U_j, f|_{U_j}) \simeq MF^\infty(U_j, f_j) \simeq \mathcal{Q}Coh^{(2)}(\mathbb{G}_m).$$

Proof. Recall that objects of $MF(U_j, f_j)$ are pairs of free finite rank vector bundles on U_j, and maps between them

$$\Bigg(\begin{tikzcd} V_1 \arrow[r, bend left, "d_1"] & V_2 \arrow[l, bend left, "d_2"] \end{tikzcd}\Bigg),$$

having the property that

$$d_1 \circ d_2 = f_j \cdot Id_{V_2}, \quad \text{and } d_2 \circ d_1 = f_j \cdot Id_{V_1}.$$

Thus the assignment

$$\left(\begin{tikzcd} V_1 \arrow[r, bend left, "d_1"] & V_2 \arrow[l, bend left, "d_2"] \end{tikzcd}\right) \in MF(U_j, f_j) \mapsto \left(\begin{tikzcd} V_1 \arrow[r, bend left, "t_j \cdot d_1"] & V_2 \arrow[l, bend left, "d_2"] \end{tikzcd}\right) \in MF(U_j, f|_{U_j})$$

determines an equivalence

$$MF(U_j, f|_{U_j}) \simeq MF(U_j, f_j).$$

The first equivalence is obtained by Ind completion.

The second equivalence follows from Knörrer periodicity. For a very general formulation of Knörrer periodicity see Theorem 9.1.7 (ii) of [P]. Let us assume for convenience that $I = \{1, 2, 3\}$, and that $j = 1$. Then $U_j = \mathbb{G}_m \times \mathbb{A}^2_\kappa$, and $f_j = t_2 \cdot t_3$, where t_2 and t_3 are coordinate on the factor $\mathbb{A}^2_\kappa$. By Knörrer periodicity $MF(U_j, f_j)$ is equivalent to the $\mathbb{Z}_2$-periodic category of perfect complexes on the first factor, $\mathbb{G}_m$. This concludes the proof. □

Remark 3.5. The equivalences constructed in Proposition 3.4 are given by explicit functors, and do not rely on further choices. For the second equivalence, this follows from the proof of Knörrer periodicity in [P]. Abusing notation we sometimes denote ι_j^* also the composition of the pull-back with the equivalences from Proposition 3.4. Thus we may write

$$\iota_j^* : MF^\infty(X_I, f) \to MF^\infty(U_j, f_j), \quad \iota_j^* : MF^\infty(X_I, f) \to \mathcal{Q}Coh^{(2)}(\mathbb{G}_m).$$

We can abstract from Remark 3.5 a formalism of restriction functors which will be useful in the next section. If L is a set of cardinality two, denote

$$X_L = Spec(\kappa[t_l, l \in L][u, u^{-1}]) \cong \mathbb{G}_m \times \mathbb{A}^2_\kappa,$$

and let f be the morphism

$$f = \times_{l \in L} t_l : X_L \longrightarrow \mathbb{A}^1_\kappa.$$

Definition 3.6. Let I and L be sets of cardinality three and two respectively, and assume that we are given an embedding $L \subset I$. We denote R^∞_{MF} the composite:

$$\begin{tikzcd} R^\infty_{MF} : MF^\infty(X_I, f) \arrow[r, "\iota_j^*"] \arrow[rr, bend right] & MF^\infty(U_j, f_j) \arrow[r, "\simeq"] & MF^\infty(X_L, f), \end{tikzcd}$$

where

1. $\{j\} = I - L$, and ι_j^* is defined as in Remark 3.5.
2. The equivalence $MF^\infty(U_j, f_j) \simeq MF^\infty(X_L, f)$ is determined by the isomorphism of κ-algebras

$$\kappa[t_i, i \in I][t_j^{-1}] = \kappa[t_l, l \in L][t_j, t_j^{-1}] \xrightarrow{\cong} \kappa[t_l, l \in L][u, u^{-1}]$$

that sends t_l to t_l, and t_j to t_u.

3.1.3. *Planar graphs and matrix factorizations.* Let G be a trivalent, planar graph. Assume for simplicity that G does not contain any loop. We will explain how to attach to G a matrix factorization-type category. We denote V_G the set of vertices of G, and E_G the set of edges.

- Let $v \in V_G$, and take a sufficiently small ball B_v in $\mathbb{R}^2$ centered at v. Then the set of connected components of $B_v - G$ has cardinality three, and we denote it I_v.
- Let $e \in V_G$, and take a sufficiently small ball B_e centered at any point in the relative interior of e. The set of connected components of $B_v - G$ has cardinality two, and we denote it L_e.

Remark 3.7. Note that the sets I_v and L_e do not depend (up to canonical identifications) on B_e and B_v. Further, if a vertex v is incident to an edge e, there is a canonical embedding: $L_e \subset I_v$.

We attach to each vertex and edge of G a category of matrix factorizations in the following way:

- We assign to $v \in V_G$ the category
$$\mathcal{B}(v) := MF^\infty(X_{I_v}, f)$$
- We assign to $e \in E_G$ the category
$$\mathcal{B}(e) := MF^\infty(X_{L_e}, f)$$

By Remark 3.7, and Definition 3.6, if a vertex v is incident to an edge e we have a restriction functor

$$R^\infty_{MF} : \mathcal{B}(v) \longrightarrow \mathcal{B}(e).$$

If two vertices v_1 and v_2 are incident to an edge e, we obtain a diagram of restriction functors

$$\mathcal{B}(v_1) \times \mathcal{B}(v_2) \rightrightarrows \mathcal{B}(e).$$

Running over the vertices and edges of G, we obtain a Čech-type diagram in $dg\mathcal{P}r^{(2),L}$

$$\textstyle\prod_{v\in V_G} \mathcal{B}(v) \rightrightarrows \prod_{e\in E_G} \mathcal{B}(e). \tag{2}$$

Definition 3.8. We denote $\mathcal{B}(G)$ the equalizer of diagram (2) in $dg\mathcal{P}r^{(2),L}$.

The definition of $\mathcal{B}(G)$ allows us to encode the category of matrix factorizations of a toric Calabi-Yau LG model in a simple combinatorial package. We use the notations of section 3.1.1.

Theorem 3.9. *Let P be a planar lattice polytope, equipped with a unimodular triangulation $\mathcal{T}$. Let $G_\mathcal{T}$ be the dual graph of $\mathcal{T}$. Then there is an equivalence in $dg\mathcal{P}r^{(2),L}$*

$$MF^\infty(X_\mathcal{T}, W_\mathcal{T}) \simeq \mathcal{B}(G_\mathcal{T}).$$

Proof. Let C be the set of maximal cones in the fan of $X_\mathcal{T}$. Consider the standard open cover of $X_\mathcal{T}$ by toric affine patches: $\{U_\sigma\}_{\sigma\in C}$, $U_\sigma \cong \mathbb{A}^3_\kappa$. By Proposition 3.3 the category $MF^\infty(X_\mathcal{T}, f_\mathcal{T})$ can be expressed as the limit of the Čech diagram for the open cover $\{U_\sigma\}_{\sigma\in C}$: the vertices of this diagram are products of the categories

$$MF^\infty(U_\sigma, f_\mathcal{T}|_{U_\sigma}), \quad \text{and} \quad MF^\infty(U_\sigma \cap U_{\sigma'}, f_\mathcal{T}|_{U_\sigma\cap U_{\sigma'}}), \quad \sigma, \sigma' \in C,$$

and the arrows are products of pullback functors.

Note that there is a natural bijection ϕ between the set $V_{\mathcal{T}}$ of vertices of $G_{\mathcal{T}}$ and C. Moreover the definition of I_v gives an identification $X_{I_v} \cong U_{\phi(\sigma)}$, and thus a canonical equivalence

$$\mathcal{B}(v) \simeq MF^{\infty}(U_{\phi(v)}, f_{\mathcal{T}}|_{\phi(v)}).$$

Similarly, by Remark 3.5, if v and v' are two vertices of $G_{\mathcal{T}}$ and e is the edge connecting them, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{B}(v) \arrow[r, "\simeq"] \arrow[d, "R^{\infty}_{MF}"'] & MF^{\infty}(U_{\phi(v)}, f_{\mathcal{T}}|_{\phi(v)}) \arrow[d, "\iota^*"] \\
\mathcal{B}(e) \arrow[r, "\simeq"] & MF^{\infty}(U_{\phi(v)} \cap U_{\phi(v')}, f_{\mathcal{T}}|_{U_{\phi(v)} \cap U_{\phi(v')}})
\end{tikzcd}$$

where the horizontal arrows are canonical equivalences, and ι^* is the restriction of matrix factorizations along the embedding

$$\iota : U_{\phi(v)} \cap U_{\phi(v')} \to U_{\phi(v)}.$$

Thus the diagrams computing $\mathcal{B}(G_{\mathcal{T}})$ and $MF^{\infty}(X_{\mathcal{T}}, f_{\mathcal{T}})$ are equivalent, and this concludes the proof. □

3.2. **A-branes.**

As explained in [HV], the mirror of a toric Calabi-Yau LG model $(X_{\mathcal{T}}, W_{\mathcal{T}})$ is a punctured Riemann surface $\Sigma_{\mathcal{T}}$ embedded as an algebraic curve in $\mathbb{C}^* \times \mathbb{C}^*$. The graph $G_{\mathcal{T}}$ is the tropicalization of $\Sigma_{\mathcal{T}}$. Since we are interested in studying the A-model on $\Sigma_{\mathcal{T}}$ we can disregard its complex structure and focus on its topology, which is captured by the genus and the number of punctures (see for instance [BS] for an explicit algebraic equation of $\Sigma_{\mathcal{T}}$). These can be read off from $G_{\mathcal{T}}$. The genus of $\Sigma_{\mathcal{T}}$ is equal to the number of relatively compact connected components in $\mathbb{R}^2 - G_{\mathcal{T}}$. The number of punctures of $\Sigma_{\mathcal{T}}$ is equal to the number of unbounded edges of $G_{\mathcal{T}}$.

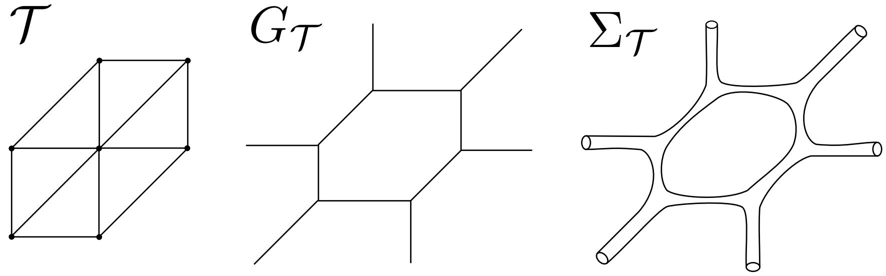

FIGURE 1. The picture shows an example of the relationship between the triangulation $\mathcal{T}$, the tropical curve $G_{\mathcal{T}}$, and the mirror curve $\Sigma_{\mathcal{T}}$.

When $\Sigma_{\mathcal{T}}$ has genus 0, the authors of [AAEKO] consider the wrapped Fukaya category of $\Sigma_{\mathcal{T}}$, and prove that it is equivalent to the category of matrix factorizations of the mirror toric LG model. In this paper we will consider an alternative model for the category of A-branes on a punctured Riemann surface, called the *topological Fukaya category.* The construction of the topological Fukaya category was first suggested by Kontsevich [K] and was studied in [DK, N1, STZ] in the case of punctured Riemann surfaces. The topological Fukaya category is conjecturally equivalent to the wrapped Fukaya category. As a byproduct of our main result we establish this equivalence in a large number of cases. We summarize the theory of the topological Fukaya category in Section 4 below.

4. The topological Fukaya category

In this Section we recall the definition of the topological Fukaya for punctured Riemann surfaces. We will mostly follow the approach of [DK], see also [STZ, N1] for related alternative formulations of this theory. It will be important to consider the *Ind*-completion of the topological Fukaya category. We discuss this in Section 4.2.

4.1. **The cyclic category and the topological Fukaya category.** We briefly review the setting of [DK]. We refer for more details to the original paper and to [D].

Definition 4.1 ([C])**.** Let Λ be the category defined as follows:

- The set of objects of Λ is in bijection with the set of natural numbers. For all $n \in N$, $\langle n \rangle \in \Lambda$ is a copy of S^1 with n marked points given by the n-th roots of unity.
- A morphism $\langle m \rangle \to \langle n \rangle$ is a homotopy class of continuous maps $S^1 \to S^1$ preserving the marked points.

Proposition 4.2. *There is a natural equivalence of categories* $(-)^* : \Lambda \to \Lambda^{op}$.

Proof. This equivalence is called *interstice duality*. See [DK] Section 2.5. ☐

Definition 4.3. Let C be a category.

- A cyclic object in C is a (contravariant) functor $X : \Lambda^{op} \to C$. We denote C_Λ the category of cyclic objects in C.
- A cocyclic object in C is a (contravariant) functor $X : \Lambda \to C$. We denote C^Λ the category of cocyclic objects in C.

Consider the map $\mathbb{A}^1_\kappa \xrightarrow{z^n} \mathbb{A}^1_\kappa$. Following [DK] we denote $\mathcal{E}^n$ the category of matrix factorizations $MF(\mathbb{A}^1_\kappa, z^n) \in dgCat^{(2)}$. The categories $\mathcal{E}^n$ assemble to a cyclic object in $dgCat^{(2)}$. We state the precise result below.

Proposition 4.4 ([DK] Proposition 2.4.1)**.** *There is a cocyclic object* $\mathcal{E}^\bullet : \Lambda \to dgCat^{(2)}$ *that is defined on objects by the assignment:*

$$\langle n \rangle \in \Lambda \mapsto \mathcal{E}^n \in dgCat^{(2)}.$$

Remark 4.5. It is often convenient to regard $\mathcal{E}^\bullet$ as a cyclic object in $(dgCat^{(2)})^{op}$.

Let $(-)^{op} : dgCat^{(2)} \to dgCat^{(2)}$ be the auto-equivalence of $dgCat^{(2)}$ sending a category to its opposite category.

Definition 4.6. Denote $\mathcal{E}_\bullet : \Lambda^{op} \to dgCat^{(2)}$ the cyclic object defined by the composition:

$$\Lambda^{op} \xrightarrow{(-)^*} \Lambda \xrightarrow{\mathcal{E}^\bullet} dgCat^{(2)} \xrightarrow{(-)^{op}} dgCat^{(2)}.$$

Remark 4.7. There are several equivalent ways to define $\mathcal{E}_\bullet$. We list them below.

(1) Denote $(-)^\vee : (dgCat^{(2)})^{op} \to dgCat^{(2)}$ the functor mapping a category D in $dgCat^{(2)}$ to $Fun(D, \mathcal{P}\mathrm{erf}^{(2)}(\kappa))$. Then $\mathcal{E}_\bullet$ is Morita equivalent to the cyclic object given by the composition:
$$\Lambda^{op} \xrightarrow{(\mathcal{E}^\bullet)^{op}} (dgCat^{(2)})^{op} \xrightarrow{(-)^\vee} dgCat^{(2)}.$$
See [DK] Section 3.2 for a discussion of this fact.

(2) All arrows in the image of the functor $\mathcal{E}^\bullet$ admit right adjoints.[2] That is, $\mathcal{E}^\bullet$ lifts to a cocyclic object in the subcategory $dgCat^{(2),L} \subset dgCat^{(2)}$. The cyclic object $\mathcal{E}_\bullet$ is Morita equivalent to the cyclic object given by the composition:
$$\Lambda^{op} \xrightarrow{(\mathcal{E}^\bullet)^{op}} (dgCat^{(2),L})^{op} \xrightarrow{\simeq} dgCat^{(2),R} \xrightarrow{\subset} dgCat^{(2)}.$$

Proposition 4.8 (Proposition 3.4.4 [DK]). *Let $\mathcal{R}ib$ be the category of ribbon graphs and contractions between them. Then there is a functor*
$$\mathcal{L} : \mathcal{R}ib \to Set_\Lambda, \quad X \in \mathcal{R}ib \mapsto \mathcal{L}(X) \in Set_\Lambda.$$

Proof. Contractions of ribbon graphs and the category $\mathcal{R}ib$ are defined [DK], to which we refer the reader also for a proof of this fact. ☐

Definition 4.9.

- Denote $\mathcal{F}^{\mathcal{E}} : Set_\Lambda \to (dgCat^{(2)})^{op}$ the homotopy right Kan extension of the diagram
$$Set_\Lambda \longleftarrow \Lambda^{op} \xrightarrow{\mathcal{E}^\bullet} (dgCat^{(2)})^{op}.$$
- Denote $\mathcal{F}_{\mathcal{E}} : Set_\Lambda \to dgCat^{(2)}$ the homotopy right Kan extension of the diagram
$$Set_\Lambda \longleftarrow \Lambda^{op} \xrightarrow{\mathcal{E}_\bullet} dgCat^{(2)}.$$

Let Σ be a Riemann surface with boundary and let $X \subset \Sigma$ be a spanning ribbon graph. The implementation of Kontsevich's ideas due to Dyckerhoff and Kapranov [DK] (see also [N1] and [STZ]) gives ways to compute a model for the Fukaya category of Σ from the combinatorics of X. We will refer to is as the topological (co)Fukaya category of X or sometimes as the topological (co)Fukaya category of the pair (Σ, X). The next definition gives the construction of the topological (co)Fukaya category, see Definition 4.1.1 of [DK].

Definition 4.10. Let (Σ, X) be a punctured Riemann surface.

- The *topological Fukaya category of* X is given by
$$\mathcal{F}^{top}(X) := \mathcal{F}^{\mathcal{E}}(\mathcal{L}(X)).$$
- The *topological coFukaya category* of X is given by
$$\mathcal{F}_{top}(X) := \mathcal{F}_{\mathcal{E}}(\mathcal{L}(X))$$

Proposition 4.11. *There is a natural equivalence*
$$\mathcal{F}_{top}(X) \simeq Fun(\mathcal{F}^{top}(X), \mathcal{P}\mathrm{erf}^{(2)}(k)).$$

Proof. See the discussion after Definition 4.1.1 of [DK]. ☐

---

[2] In fact, they admit also left adjoints.

Remark 4.12. We point out that in [DK] the authors give a different definition of the topological Fukaya category. In their formulation our Definition 4.10 is a theorem, see Theorem 4.1.2 of [DK].

4.2. **The Ind-completion of the topological Fukaya category.** In this Section we introduce the Ind-completed version of the topological (co)Fukaya category. This plays an important role in proving that $\mathcal{F}^{top}(-)$ exhibits an interesting sheaf-like behavior with respect to closed coverings of ribbon graphs, for which see Section 5.2. We remark that Definitions 4.13, 4.14 and 4.15 mirror exactly Proposition 4.4 and Definition 4.6, 4.9 and 4.10 from the previous Section, the only difference being that we are now working with presentable dg-categories.

Definition 4.13.

- Denote $\mathcal{IE}^{\bullet} : \Lambda^{op} \to dg\mathcal{P}r^{(2),L}$ the co-cyclic object defined by the composition:
$$\Lambda \xrightarrow{\mathcal{E}^{\bullet}} dgCat^{(2)} \xrightarrow{Ind} dg\mathcal{P}r^{(2),L}.$$
- Denote $\mathcal{IE}_{\bullet} : \Lambda^{op} \to dg\mathcal{P}r^{(2),L}$ the cyclic object defined by the composition:
$$\Lambda^{op} \xrightarrow{\mathcal{E}_{\bullet}} dgCat^{(2)} \xrightarrow{Ind} dg\mathcal{P}r^{(2),L}.$$

Definition 4.14.

- Denote $\mathcal{IF}^{\mathcal{E}} : Set_{\Lambda} \to (dg\mathcal{P}r^{(2),L})^{op} \simeq dg\mathcal{P}r^{(2),R}$ the homotopy right Kan extension of the diagram
$$Set_{\Lambda} \longleftarrow \Lambda^{op} \xrightarrow{\mathcal{IE}^{\bullet}} (dg\mathcal{P}r^{(2),L})^{op} \xrightarrow{\simeq} dg\mathcal{P}r^{(2),R}.$$
- Denote $\mathcal{IF}_{\mathcal{E}} : Set_{\Lambda} \to (dg\mathcal{P}r^{(2),L})^{op}$ the homotopy right Kan extension of the diagram
$$Set_{\Lambda} \longleftarrow \Lambda^{op} \xrightarrow{\mathcal{IE}_{\bullet}} dg\mathcal{P}r^{(2),L}.$$

Definition 4.15. Let X be a ribbon graph. We make the following notations:

- $\mathcal{IF}^{top}(X) := \mathcal{IF}^{\mathcal{E}}(\mathcal{L}(X))$.
- $\mathcal{IF}_{top}(X) := \mathcal{IF}_{\mathcal{E}}(\mathcal{L}(X))$.

Proposition 4.16. *Let X be a ribbon graph. Then $\mathcal{IF}^{top}(X)$ is equivalent to $Ind(\mathcal{F}^{top}(X))$.*

Proof. This follows from the fact that *Ind*-completion preserves homotopy colimits. ☐

Remark 4.17. The category $\mathcal{IF}_{top}(X)$ is not equivalent to the Ind-completion $Ind(\mathcal{F}_{top}(X))$. Instead $Ind(\mathcal{F}_{top}(X))$ is a full subcategory of $\mathcal{IF}_{top}(X)$. We clarify the relationship between these two categories in Proposition 4.19 and Example 4.20 below.

Lemma 4.18. *Let $F : A \to B$ be in a functor $dgCat^{(2)}$. Assume that $G : B \to A$ is the right adjoint of F. Then $Ind(G) : Ind(A) \to Ind(B)$ is the right adjoint of $Ind(F)$.*

Let i_R and i_L be the inclusions of $dg\mathcal{P}r^{(2),R}$ and $dg\mathcal{P}r^{(2),L}$ into $\widehat{dgCat^{(2)}}$. Let $\widehat{\mathcal{IF}^{\mathcal{E}}}$ and $\widehat{\mathcal{IF}_{\mathcal{E}}}$ be the compositions of the functors

$$Set_{\Lambda} \xrightarrow{\mathcal{IF}^{\mathcal{E}}} dg\mathcal{P}r^{(2),R} \xrightarrow{i_R} \widehat{dgCat^{(2)}}, \quad Set_{\Lambda} \xrightarrow{\mathcal{IF}_{\mathcal{E}}} dg\mathcal{P}r^{(2),L} \xrightarrow{i_L} \widehat{dgCat^{(2)}}.$$

Proposition 4.19. *The functors*

$$\widehat{\mathcal{IF}^{\mathcal{E}}}, \widehat{\mathcal{IF}_{\mathcal{E}}} : Set_\Lambda \to \widehat{dgCat^{(2)}}$$

are naturally equivalent.

Proof. Recall that the isomorphism $(dg\mathcal{P}r^{(2),L})^{op} \simeq dg\mathcal{P}r^{(2),R}$ sends an arrow in $(dg\mathcal{P}r^{(2),L})^{op}$ to its right adjoint. Informally, this implies that if we regard $\mathcal{IE}^\bullet$ and $\mathcal{IE}_\bullet$ just as diagrams of presentable dg-categories and functors between them, then they are Morita equivalent. Let us clarify this statement. Recall that by Definition 4.15 the cyclic object $\mathcal{IE}^\bullet$ is given by

$$\Lambda \xrightarrow{\mathcal{E}^\bullet} (dgCat^{(2)})^{op} \xrightarrow{Ind} (dg\mathcal{P}r^{(2),L})^{op} \xrightarrow{\simeq} dg\mathcal{P}r^{(2),R}.$$

By Remark 4.7 the cyclic object $\mathcal{IE}_\bullet$ is Morita equivalent to the composition

$$\Lambda^{op} \xrightarrow{(\mathcal{E}^\bullet)^{op}} (dgCat^{(2),L})^{op} \xrightarrow{\simeq} dgCat^{(2),R} \xrightarrow{\subset} dgCat^{(2)} \xrightarrow{Ind} dg\mathcal{P}r^{(2),L}.$$

Next consider the composition of $\mathcal{IE}^\bullet$ and $\mathcal{IE}_\bullet$ with i_R and i_L. We denote $\widehat{\mathcal{IE}^\bullet}$ and $\widehat{\mathcal{IE}_\bullet}$ the resulting cyclic objects in $\widehat{dgCat^{(2)}}$. The claim that $\widehat{\mathcal{IE}^\bullet}$ and $\widehat{\mathcal{IE}_\bullet}$ are Morita equivalent follows immediately from Lemma 4.18: indeed, by the Lemma, taking Ind completions commutes with taking right adjoints.

As a consequence $\widehat{\mathcal{IF}^{\mathcal{E}}}$ and $\widehat{\mathcal{IF}_{\mathcal{E}}}$ are calculated by taking the limit of the same diagram, except the limit takes place in two different ambient categories. Namely by the general formula for Kan extensions, we have that:

1. The functor $\widehat{\mathcal{IF}^{\mathcal{E}}}(-) : Set_\Lambda \to \widehat{dgCat^{(2)}}$ is obtained by computing the homotopy limit
$$\mathcal{L} \in Set_\Lambda \mapsto holim_{\{L(\langle n\rangle)\to\mathcal{L}\}}\mathcal{IE}_n,$$
in the category $dg\mathcal{P}r^{(2),R}$, and then applying $i_R : dg\mathcal{P}r^{(2),R} \to \widehat{dgCat^{(2)}}$.
2. The functor $\widehat{\mathcal{IF}_{\mathcal{E}}}(-) : Set_\Lambda \to \widehat{dgCat^{(2)}}$ is obtained by computing the homotopy limit
$$\mathcal{L} \in Set_\Lambda \mapsto holim_{\{L(\langle n\rangle)\to\mathcal{L}\}}\mathcal{IE}_n,$$
in the category $dg\mathcal{P}r^{(2),L}$, and then applying $i_L : dg\mathcal{P}r^{(2),R} \to \widehat{dgCat^{(2)}}$.

By Proposition 5.5.3.13 and 5.5.3.18 of [Lu] i_L and i_R preserve finite limits. Thus $\widehat{\mathcal{IF}^\bullet}$ and $\widehat{\mathcal{IF}_\bullet}$ can described in an equivalent way as the homotopy Kan extension of the cyclic objects $\widehat{\mathcal{IE}^\bullet}$ and $\widehat{\mathcal{IE}_\bullet}$ in $\widehat{dgCat^{(2)}}$. As we discussed $\widehat{\mathcal{IE}^\bullet}$ and $\widehat{\mathcal{IE}_\bullet}$ are Morita equivalent, and therefore their Kan extensions are equivalent as well. This completes the proof. ☐

Example 4.20. Consider the ribbon graph X given by a loop with no vertices. We can tabulate the value of the topological (co)Fukaya category of X and of its Ind-completed version as follows:

- The category $\mathcal{F}^{top}(X)$ is equivalent to $\mathcal{P}\mathrm{erf}^{(2)}(\mathbb{G}_m)$.
- The category $\mathcal{F}_{top}(X)$ is equivalent to the full subcategory of $\mathcal{P}\mathrm{erf}^{(2)}(\mathbb{G}_m)$ given by complexes that have compact support, $\mathcal{P}\mathrm{erf}^{(2)}_{cs}(\mathbb{G}_m)$.
- By Proposition 4.19 the categories $\mathcal{IF}^{top}(X)$ and $\mathcal{IF}_{top}(X)$ are both equivalent to $\mathcal{Q}Coh^{(2)}(\mathbb{G}_m)$.

Note that $Ind(\mathcal{F}_{top}(X)) \simeq Ind(\mathcal{P}\mathrm{erf}^{(2)}_{cs}(\mathbb{G}_m))$ is a strict sub-category of $\mathcal{IF}_{top}(X) \simeq \mathcal{Q}Coh(\mathbb{G}_m)$ (see Remark 4.17).

Definition 4.21.
- We denote $\mathcal{E}^{\bullet}_{\infty}$ the cyclic object $\widehat{\mathcal{IE}^{\bullet}} \simeq \widehat{\mathcal{IE}_{\bullet}}$ in $\widehat{dgCat^{(2)}}$.
- We denote $\mathcal{F}^{\mathcal{E}}_{\infty}$ the functor $\widehat{\mathcal{IF}^{\mathcal{E}}} \simeq \widehat{\mathcal{IF}_{\mathcal{E}}} : Set_{\Lambda} \to \widehat{dgCat^{(2)}}$.
- We denote $\mathcal{F}^{top}_{\infty}$ the functor $\mathcal{F}^{\mathcal{E}}_{\infty} \circ \mathcal{L} : \mathcal{R}ib \to \widehat{dgCat^{(2)}}$. If X is a ribbon graph, we call $\mathcal{F}^{top}_{\infty}(X)$ the *Ind-complete topological Fukaya category of X*.

5. The topological Fukaya category and restrictions

In this Section we explore various naturality properties of $\mathcal{F}^{top}_{\infty}$ with respect to open and closed embeddings of ribbon graphs. As first suggested by Kontsevich [K], the topological Fukaya category behaves like a (co)sheaf with respect to open covers. This aspect was investigated in [DK, STZ]. One of our main theorems is that, additionally, the topological Fukaya category behaves like a sheaf also with respect to certain closed covers.

5.1. **Restriction to open subgraphs.** Let X be a ribbon graph. With small abuse of notation we denote X also its underlying topological space. We say that $Y \subset X$ is a *subgraph* if it is a subspace having the property that, if the of Y with an edge e of X is not empty or a vetex, then e is contained in Y. If Y is a subgraph of X, then it has a canonical structure of ribbon graph. Note that if U and V are open subgraphs of X, then their intersection $U \cap V$ is also an open subgraph of X.

Proposition 5.1. *Let X be a ribbon graph and let $U \subset X$ be an open subgraph. Then there are corestriction functors*
- $C_U : \mathcal{F}^{top}(U) \to \mathcal{F}^{top}(X)$
- $C_{\infty,U} : \mathcal{F}^{top}_{\infty}(U) \to \mathcal{F}^{top}_{\infty}(X)$

When the target of the corestriction functors is not clear from the context we will use the notations C^X_U and $C^X_{\infty,U}$.

Proof. We refer to [D] Section 4 for a treatment of the corestriction functor C_U. The functor $C_{\infty,U}$ is obtained by Ind-completion. ☐

Remark 5.2. By Proposition 4.16 $\mathcal{F}^{top}_{\infty}(U)$ and $\mathcal{F}^{top}_{\infty}(X)$ are equivalent to the Ind-completions of $\mathcal{F}^{top}(U)$ and $\mathcal{F}^{top}(X)$. There is a natural equivalence

$$C_{\infty,U} \simeq Ind(C_U) : \mathcal{F}^{top}_{\infty}(U) \to \mathcal{F}^{top}_{\infty}(X).$$

In particular the corestriction $C_{\infty,U}$ is a morphism in $dg\mathcal{P}r^{(2),L}$.

Definition 5.3. Let X be a ribbon graph and let $U \subset X$ be an open subgraph. Then we define *restriction functors*:
- By Proposition 4.11 there is a natural equivalence between $\mathcal{F}_{top}(-)$ and the dual of $\mathcal{F}^{top}(-)$. The restriction $R^U : \mathcal{F}_{top}(X) \to \mathcal{F}_{top}(U)$ is obtained by dualizing C_U.
- $R^U_{\infty} : \mathcal{F}^{top}_{\infty}(X) \to \mathcal{F}^{top}_{\infty}(U)$ is the right adjoint of the corestriction $C_{\infty,U}$.

When the target of the restriction functors is not clear from the context we will use the notations R^U_X and $R^U_{\infty,X}$.

Remark 5.4. Note that the functor R^U_∞ cannot be realized as $Ind(R^U)$. As we explain in Remark 4.17, in general, the functors R^U_∞ and $Ind(R^U)$ have different source and target.

Remark 5.5. Let X be a ribbon graph. Let $V \subset U \subset X$ be open subgraphs. Then the following diagram commutes:

$$\begin{tikzcd} & \mathcal{F}^{top}_\infty(U) \arrow[dr, "R^V_\infty"] & \\ \mathcal{F}^{top}_\infty(X) \arrow[ur, "R^U_\infty"] \arrow[rr, "R^V_\infty"] & & \mathcal{F}^{top}_\infty(V). \end{tikzcd}$$

Proposition 5.6. *Let X be a ribbon graph and let U and V be open subgraphs such that $X = U \cup V$.*

- *The following is a homotopy push-out in $dgCat^{(2)}$:*

$$\begin{tikzcd} \mathcal{F}^{top}(U \cap V) \arrow[r, "C_{U\cap V}"] \arrow[d, "C_{U\cap V}"'] & \mathcal{F}^{top}(U) \arrow[d, "C_U"] \\ \mathcal{F}^{top}(V) \arrow[r, "C_V"] & \mathcal{F}^{top}(X). \end{tikzcd}$$

- *The following is a homotopy push-out in $dg\mathcal{P}r^{(2),L}$:*

$$\begin{tikzcd} \mathcal{F}^{top}_\infty(U \cap V) \arrow[r, "C_{\infty,U\cap V}"] \arrow[d, "C_{\infty,U\cap V}"'] & \mathcal{F}^{top}_\infty(U) \arrow[d, "C_{\infty,U}"] \\ \mathcal{F}^{top}_\infty(V) \arrow[r, "C_{\infty,V}"] & \mathcal{F}^{top}_\infty(X). \end{tikzcd}$$

Proof. The first part of the claim can be proved in the same way as Proposition 4.2 of [D]. The second part follows because the Ind-completion commutes with colimits. □

Proposition 5.7. *Let X be a ribbon graph and let U and V be open subgraphs such that $X = U \cup V$.*

- *The following is a homotopy fiber product in $dgCat^{(2)}$:*

$$\begin{tikzcd} \mathcal{F}_{top}(X) \arrow[r, "R^U"] \arrow[d, "R^V"'] & \mathcal{F}_{top}(U) \arrow[d, "R^{U\cap V}"] \\ \mathcal{F}_{top}(V) \arrow[r, "R^{U\cap V}"] & \mathcal{F}_{top}(U \cap V). \end{tikzcd}$$

- *The following is a homotopy fiber product in $\widehat{dgCat}^{(2)}$:*

$$\begin{tikzcd} \mathcal{F}^{top}_\infty(X) \arrow[r, "R^U_\infty"] \arrow[d, "R^V_\infty"'] & \mathcal{F}^{top}_\infty(U) \arrow[d, "R^{U\cap V}_\infty"] \\ \mathcal{F}^{top}_\infty(V) \arrow[r, "R^{U\cap V}_\infty"] & \mathcal{F}^{top}_\infty(U \cap V). \end{tikzcd}$$

Proof. The claim follows by dualizing the push-outs in Proposition 5.6. □

Remark 5.8. The second diagram of Proposition 5.7 has very different formal properties from the second diagram of Proposition 5.6. If $U \subset X$ is an open subgraph the corestriction $C_{\infty,X}$ preserves compact objects , but in general this is not the case for the restriction R^U_∞ (see Remark 5.4). Thus (in general) we cannot apply $(-)^\omega$ to the second diagram of Proposition 5.7 and obtain a diagram of small categories.

Proposition 5.6 and Proposition 5.7 can be extended in the usual way to account for arbitrary open covers of X: given any open cover of X, the (Ind-completed) (co)Fukaya category can be realized as the homotopy (co)limit of the appropriate Čech diagram of local sections. This clarifies that this formalism is indeed an implementation of the idea that the Fukaya category of a punctured surface should define either a sheaf or a cosheaf of categories on its spine.

5.2. **Restriction to closed subgraphs.** In this Section we turn our attention to closed subgraphs and closed covers of ribbon graphs. In the context of the topological Fukaya category, restrictions to closed subgraphs have also been investigated by Dyckerhoff [D]. To avoid producing here parallel arguments we will refer to the lucid treatment contained in Section 4 of [D].

Definition 5.9. Let X be a ribbon graph.

- An open subgraph U of X is *good* if its complement $X - U$ does not have vertices of valency one.
- A closed subgraph Z of X is *good* if it is the complement in X of a good open subgraph.

We introduce next restriction functors of $\mathcal{F}^{top}$ and $\mathcal{F}^{top}_\infty$ to good closed subgraphs: we will sometimes refer to these as *exceptional restrictions*, in order to mark their difference from the (co)restrictions to open subgraphs that were discussed in the previous Section.

Proposition 5.10. *Let X be a ribbon graph and let $Z \subset X$ be a good closed subgraph. Then there are* exceptional restriction *functors*

- $S^Z : \mathcal{F}^{top}(X) \to \mathcal{F}^{top}(Z)$
- $S^Z_\infty : \mathcal{F}^{top}_\infty(X) \to \mathcal{F}^{top}_\infty(Z)$.

When the source of the exceptional restriction functors is not clear from the context we will use the notations S^Z_X and $S^Z_{\infty,X}$.

Proof. For a definition of S^Z see Section 4 of [D]. The functor S^Z_∞ is the Ind completion of S^Z. □

Remark 5.11. The property that the closed subgraph Z is good is not strictly necessary to define exceptional restrictions. However this assumption allows for a somewhat simpler exposition, and it is essential in Theorem 6.5 in the next section. We refer the reader to [D] for a treatment of exceptional restrictions which does not impose additional requirements on the closed subgraphs.

Proposition 5.12. *Let X be a ribbon graph, let $U \subset X$ be a good open subset and let $Z = X - U$. The following are cofiber sequences in $dgCat^{(2)}$ and in $dg\mathcal{P}r^{(2),L}$ respectively:*

$$\mathcal{F}^{top}(U) \overset{C_X}{\to} \mathcal{F}^{top}(X) \overset{R^Z}{\to} \mathcal{F}^{top}(Z), \quad \mathcal{F}^{top}_\infty(U) \overset{C_{\infty,X}}{\to} \mathcal{F}^{top}_\infty(X) \overset{R^Z_\infty}{\to} \mathcal{F}^{top}_\infty(Z)$$

Proof. See Proposition 4.9 of [D]. □

Definition 5.13. Let X be a ribbon graph and let $Z \subset X$ be a good closed subgraph. Then we denote

$$T^X_\infty : \mathcal{F}^{top}_\infty(Z) \to \mathcal{F}^{top}_\infty(X)$$

the right adjoint of the exceptional restriction functor. We will call it the *exceptional corestriction* functor. When the source of the exceptional corestriction functor is not clear from the context we will use the notation $T^X_{\infty,Z}$.

Proposition 5.14 is a compatibility statement that relates the various restrictions that we have introduced so far, and it will be useful in the next section.

Proposition 5.14. *Let X be a ribbon graph. Let $U \subset X$ be an open subgraph and let $Z \subset X$ be a good closed subgraph. If Z is contained in U then the following diagram commutes:*

$$\begin{tikzcd} & \mathcal{F}^{top}_\infty(U) \arrow[dr, "S^Z_{\infty,U}"] & \\ \mathcal{F}^{top}_\infty(X) \arrow[ur, "R^U_\infty"] \arrow[rr, "S^Z_{\infty,X}"] & & \mathcal{F}^{top}_\infty(Z). \end{tikzcd}$$

Before proving Proposition 5.14 we introduce some preliminary notations and results.

Definition 5.15. Let x be a vertex of X. We denote:

- U_x, the open subgraph of X containing only the vertex x
- K_x, the open subgraph of X given by $X - x$
- U^p_x, the intersection $U_x \cap K_x$ (the superscript p stands for *punctured* neighborhood)

Definition 5.16. Let $F : A \to \mathcal{F}^{top}_\infty(K_x)$ be a fully-faithful functor. We say that A is *not supported on x* if the composite

$$A \xrightarrow{F} \mathcal{F}^{top}_\infty(K_x) \xrightarrow{R^{U^p_x}_\infty} \mathcal{F}^{top}_\infty(U^p_x)$$

is the zero functor.

Lemma 5.17. *Let x be a vertex of X, and let $F : A \to \mathcal{F}^{top}_\infty(K_x)$ be a fully-faithful functor. Denote $F : A \to \mathcal{F}^{top}_\infty(K_x)$ the inclusion. If A is not supported on x, then there is a natural equivalence*

$$R^{K_x}_{\infty,X} \circ C^X_{\infty,K_x} \circ F \simeq F.$$

Proof. We fix first some conventions. If Γ is a ribbon graph and W_Γ is a subset of the vertices of Γ, we set

$$\Gamma(W_\Gamma) := \coprod_{v \in V_\Gamma} \Gamma_v, \quad \Gamma^2(V_\Gamma) := \coprod_{v,v' \in W_\Gamma} \Gamma_v \cap \Gamma_{v'}.$$

Also if L and L' are objects in $\mathcal{F}^{top}_\infty(\Gamma)$, we denote their Hom-complex $Hom_\Gamma(L, L')$.

Let us now return to the statement of the lemma. By Proposition 5.7 the Ind-complete topological Fukaya category $\mathcal{F}^{top}_\infty(X)$ is naturally equivalent to the equalizer

$$\mathcal{F}^{top}_\infty(X) \xrightarrow{\prod R^{X_v}_\infty} \Big(\prod\nolimits_{v \in V} \mathcal{F}^{top}_\infty(X_v) \overset{r_1}{\underset{r_2}{\rightrightarrows}} \prod\nolimits_{v_1,v_2 \in V} \mathcal{F}^{top}_\infty(X_{v_1} \cap X_{v_2}) \Big) \tag{3}$$

in $\widehat{dgCat^{(2)}}$, where r_1 and r_2 are products of restriction functors. Note that, to simplify notations, we can write

$$\prod_{v \in V} \mathcal{F}^{top}_{\infty}(X_v) \simeq \mathcal{F}^{top}_{\infty}(X(V)), \text{ and } \prod_{v_1, v_2 \in V} \mathcal{F}^{top}_{\infty}(X_{v_1} \cap X_{v_2}) \simeq \mathcal{F}^{top}_{\infty}(X^2(V)).$$

Similarly if $W = V - \{x\}$ is the set of vertices of K_x, we obtain an equalizer diagram[3]

$$\begin{tikzcd} \mathcal{F}^{top}_{\infty}(K_x) \arrow[r, "\prod R^{X_v}_{\infty}"] & \Big(\mathcal{F}^{top}_{\infty}(X(W)) \arrow[r, shift left, "r'_2"] \arrow[r, shift right, "r'_1"'] & \mathcal{F}^{top}_{\infty}(X^2(W))\Big). \end{tikzcd} \tag{4}$$

The inclusion $W \subset V$ gives projections P and Q that fit in a morphism of diagrams

$$\begin{tikzcd} \Big(\mathcal{F}^{top}_{\infty}(X(V)) \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "P"'] & \mathcal{F}^{top}_{\infty}(X^2(V))\Big) \arrow[d, "Q"] \\ \Big(\mathcal{F}^{top}_{\infty}(X(W)) \arrow[r, shift left, "r'_1"] \arrow[r, shift right, "r'_2"'] & \mathcal{F}^{top}_{\infty}(X^2(W))\Big). \end{tikzcd} \tag{5}$$

Further, the restriction $R^{K_x}_{\infty, X}$ coincides with the morphism between the equalizers $\mathcal{F}^{top}_{\infty}(X)$ and $\mathcal{F}^{top}_{\infty}(K_x)$ induced by (1). Denote $(P)^L$ and $(Q)^L$ the left adjoints of P and Q. The functor $(P)^L$ is given by the obvious fully-faithful embedding

$$\mathcal{F}^{top}_{\infty}(X(W)) \xrightarrow{\subset} \mathcal{F}^{top}_{\infty}(X(V)) \simeq \mathcal{F}^{top}_{\infty}(X(W)) \times \mathcal{F}^{top}_{\infty}(U_x),$$

and similarly for $(Q)^L$.

We will prove the lemma in two steps. First, we show that the diagram

$$\begin{tikzcd} & \Big(\mathcal{F}^{top}_{\infty}(X(V)) \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & \mathcal{F}^{top}_{\infty}(X^2(V))\Big) \\ A \arrow[r, "F"] & \Big(\mathcal{F}^{top}_{\infty}(X(W)) \arrow[r, shift left, "r'_1"] \arrow[r, shift right, "r'_2"'] \arrow[u, "(P)^L"] & \mathcal{F}^{top}_{\infty}(X^2(W))\Big) \arrow[u, "(Q)^L"'] \end{tikzcd} \tag{6}$$

is commutative, in the sense that there are natural equivalences

$$r_1 \circ (P)^L \circ F \simeq (Q)^L \circ r'_1 \circ F, \text{ and } r_2 \circ (P)^L \circ F \simeq (Q)^L \circ r'_2 \circ F.$$

We remark that the commutativity does not hold if we do not precompose with F. It is sufficient to prove that, for all v and v' in V, the diagram commutes after composing on the left with the restriction

$$R^{U_v \cap U_{v'}}_{\infty} : \mathcal{F}^{top}_{\infty}(X^2(V)) \to \mathcal{F}^{top}_{\infty}(U_v \cap U_{v'}).$$

If both v and v' are different from x, then

$$R^{U_v \cap U_{v'}}_{\infty} \circ r_i \circ (P)^L \simeq R^{U_v \cap U_{v'}}_{\infty} \circ (Q)^L \circ r'_i,$$

---

[3] In fact this holds only if the valency of x is greater than two, which we shall assume. The proof in the case when x has valency two is similar.

and so commutativity holds also when precomposing with F. Assume on the other hand that $v = x$. Then

$$R_{\infty}^{U_x \cap U_{v'}} \circ r_i \circ (P)^L \circ F \simeq 0 \simeq R_{\infty}^{U_x \cap U_{v'}} \circ (Q)^L \circ r_i' \circ F.$$

The first equivalence follows from the support assumption on A, and the second one is a consequence of the fact that $R_{\infty}^{U_x \cap U_{v'}} \circ (Q)^L = 0$. Thus diagram (6) commutes as claimed.

The commutativity of (6), and the universal property of the equalizer, give us a functor

$$\tilde{F} : A \longrightarrow \mathcal{F}_{\infty}^{top}(X).$$

Note that both $P \circ (P)^L$ and $Q \circ (Q)^L$ are naturally equivalent to the identity functor, and thus

$$R_{\infty,X}^{K_x} \circ \tilde{F} \simeq F.$$

The second and final step in the proof consists in noticing that $\tilde{F}$ is equivalent to $C_{\infty,K_x}^{X} \circ F$. That is, for all L_A in the image of F, and L_X in $\mathcal{F}_{\infty}^{top}(X)$, there is a natural equivalence

$$Hom_X(\tilde{F}(L_A), L_X) \simeq Hom_{K_x}(L_A, R_{\infty,X}^{K_x}(L_X)),$$

where $Hom_X(-,-)$ and $Hom_{K_x}(-,-)$ denote respectively the hom spaces in $\mathcal{F}_{\infty}^{top}(X)$ and in $\mathcal{F}_{\infty}^{top}(K_x)$. This can be checked by computing explicitly the Hom-complexes on both sides in terms of the equalizers (3) and (4), see Proposition 2.2 of [STZ] for a similar calculation. As a consequence there is a chain of equivalences

$$R_{\infty,X}^{K_x} \circ C_{\infty,K_x}^{X} \circ F \simeq R_{\infty,X}^{K_x} \circ \tilde{F} \simeq F$$

and this concludes the proof. □

Let Z be a good closed subgraph of X. Recall that

$$T_{\infty,Z}^{X} : \mathcal{F}_{\infty}^{top}(Z) \longrightarrow \mathcal{F}_{\infty}^{top}(X)$$

is the right adjoint of $S_{\infty,Z}^{X}$.

Proposition 5.18. *Let x be a vertex of X which does not belong to Z. Then*

1. *The vertex x does not lie in the support of*

$$\mathcal{F}_{\infty}^{top}(Z) \xrightarrow{T_{\infty,Z}^{K_x}} \mathcal{F}_{\infty}^{top}(K_x).$$

2. *The diagram*

$$\begin{tikzcd}
 & \mathcal{F}_{\infty}^{top}(K_x) \arrow[dl, "C_{\infty,K_x}^{X}"'] & \\
\mathcal{F}_{\infty}^{top}(X) & & \mathcal{F}_{\infty}^{top}(Z) \arrow[ul, "T_{\infty,Z}^{K_x}"'] \arrow[ll, "T_{\infty,Z}^{X}"]
\end{tikzcd}$$

is commutative.

Proof. It follows from Proposition 5.12 that

$$\mathcal{F}_{\infty}^{top}(Z) \xrightarrow{T_{\infty,Z}^{K_x}} \mathcal{F}_{\infty}^{top}(K_x) \xrightarrow{R_{\infty,K_x}^{K_x - Z}} \mathcal{F}_{\infty}^{top}(K_x - Z)$$

is a fiber sequence, and thus the composite is the zero functor. Since U_x^p is contained in $K_x - Z$, the restriction $R^{U_x^p}_{\infty,K_x}$ factors through $R^{K_x-Z}_{\infty,K_x}$. This implies the first claim. As for the second claim, let us show first that there is a natural equivalence

$$T^{K_x}_{\infty,Z} \simeq R^{K_x}_{\infty,X} \circ T^{X}_{\infty,Z}. \tag{7}$$

Consider the commutative square on the right hand side of the following diagram

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(Z) \arrow[r, "T^{X}_{\infty,Z}"] \arrow[d, dashed] & \mathcal{F}^{top}_{\infty}(X) \arrow[r, "R^{X-Z}_{\infty,X}"] \arrow[d, "R^{K_x}_{\infty,X}"'] & \mathcal{F}^{top}(X-Z) \arrow[d, "R^{K_x-Z}_{\infty,X}"] \\
\mathcal{F}^{top}_{\infty}(Z) \arrow[r, "T^{K_x}_{\infty,Z}"] & \mathcal{F}^{top}_{\infty}(K_x) \arrow[r, "R^{K_x-Z}_{\infty,K_x}"] & \mathcal{F}^{top}_{\infty}(K_x-Z).
\end{tikzcd}$$

The functor induced between the homotopy fibers, which is denoted by the dashed arrow, is equivalent to the identity. This gives equivalence (7). As a consequence we obtain

$$C^{X}_{\infty,K_x} \circ T^{K_x}_{\infty,Z} \simeq C^{X}_{\infty,K_x} \circ R^{K_x}_{\infty,X} \circ T^{X}_{\infty,Z} \simeq T^{X}_{\infty,Z}.$$

Indeed, the first equivalence follows from (7) and the second one from Lemma 5.17. This concludes the proof. ☐

The proof of Proposition 5.14. Let W be the set of vertices of X that do not belong to U. Note that U is a connected component of the open subgraph $(X - W) \subset X$. By induction it is sufficient to prove the claim in the following two cases:

1. when U is equal to K_x for some vertex x of X, and
2. when U is a connected component of X.

Proposition 5.18 gives a proof in the first case. Indeed, it is sufficient to take right adjoints in Claim (2) of Proposition 5.18 to recover the commutativity statement from Proposition 5.14 for this class of open subgraphs. The second case is easier. The complement $X - U$ is open and we have a splitting

$$\mathcal{F}^{top}_{\infty}(X) \simeq \mathcal{F}^{top}_{\infty}(U) \times \mathcal{F}^{top}_{\infty}(X-U),$$

and the claim follows immediately from here. ☐

6. The topological Fukaya category and closed covers

In this section we prove a key gluing statement which is one of the main inputs in our proof of mirror symmetry for three dimensional LG models.

Let X be a connected ribbon graph whose underlying topological space is homeomorphic to a copy of S^1 together a finite number of open edges attached to it. We call such a ribbon graph a *wheel*. Any choice of orientation on S^1 partitions the sets of open edges of X into two subsets, which we call *upward* and *downward edges* respectively. For our purposes it will not be important to be able to tell which of the two subsets are the upward and which the downward edges, but only to distinguish between the two. Thus we do not need to impose on X any additional structure beyond the ribbon structure (such as a choice of orientation).

Definition 6.1. Let n_1 and n_2 be in $\mathbb{Z}_{\geq 0}$. We denote $\Lambda(n_1, n_2)$ a wheel with n_1 upward and n_2 downward edges.

The underlying topological space of $\Lambda(0,0)$ is S^1, possibly equipped with a collection of 2-valent vertices. We denote $E(+)$ and $E(-)$ the open subgraphs of $\Lambda(n_1, n_2)$ given by the collection of the n_1 upward edges, and of the n_2 downward edges respectively.

Remark 6.2. The notation $\Lambda(n_1, n_2)$ does not pick out a single ribbon graph, but rather a class of ribbon graphs. Indeed specifying the number of upward and downward edges does not suffice to pin down a homeomorphism type, or even the number of vertices. However all graphs of type $\Lambda(n_1, n_2)$ deform into one another in a way that does not affect the sections of $\mathcal{F}^{top}$ and $\mathcal{F}^{top}_{\infty}$ (see [DK]). We use $\Lambda(n_1, n_2)$ to refer to any ribbon graph having the properties listed in the definition.

The category $\mathcal{F}^{top}_{\infty}(\Lambda(n_1, n_2))$ can be described explicitly in terms of quiver representations. Consider the closed subgraph

$$S \subset \Lambda(n_1, n_2)$$

where S is the central circle of the wheel. The graph S has $n_1 + n_2$ bivalent vertices, which are in canonical bijection with the spokes of $\Lambda(n_1, n_2)$, and its underlying topological space is S^1. Label the vertices of S with $+$ or $-$ depending on whether the corresponding spoke is upward or downward. We choose an orientation on S. An orientation determines a cyclic order on the edges of S. If e is edge we denote $\tau(e)$ the edge that follows it in the cyclic order. There is a (unique) vertex of S incident to both e and $\tau(e)$: we say that the pair $e, \tau(e)$ is right-handed if this vertex is labeled by a $+$, and left-handed if it is labeled by a $-$.

Let $Q(n_1, n_2)$ be the quiver defined as follows

- The set of vertices of $Q(n_1, n_2)$ is the set the edges of S. If e is an edge of S, we denote v_e the corresponding vertex of $Q(n_1, n_2)$.
- There is an arrow joining v_e and $v_{\tau(e)}$. It is oriented from v_e to $v_{\tau(e)}$ if the pair $e, \tau(e)$ is right-handed, and from $v_{\tau(e)}$ to v_e otherwise.

Denote $Rep^{\infty}(Q(n_1, n_2))$ the triangulated dg category of (non-necessarily finite dimensional) representations of $Q(n_1, n_2)$.

Lemma 6.3. *There is an equivalence*

$$\mathcal{F}^{top}_{\infty}(\Lambda(n_1, n_2)) \simeq Rep^{\infty}(Q(n_1, n_2)).$$

Proof. This is proved as Theorem 3.7 in [STZ]. ☐

Assume that we have closed embeddings

$$\Lambda(n_1, n_2) \subset \Lambda(m_1, m_2), \quad \Lambda(n'_1, n'_2) \subset \Lambda(m_1, m_2),$$

and that $\Lambda(n_1, n_2) \cup \Lambda(n'_1, n'_2) = \Lambda(m_1, m_2)$. The intersection of $\Lambda(n_1, n_2)$ and $\Lambda(n'_1, n'_2)$ is also a wheel-type graph, which we denote $\Lambda(l_1, l_2)$.

Lemma 6.4. *The following is a fiber product in* $\widehat{dgCat}^{(2)}$

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(\Lambda(m_1, m_2)) \arrow[r, "S^{\Lambda(n_1,n_2)}_{\infty}"] \arrow[d, "S^{\Lambda(n'_1,n'_2)}_{\infty}"'] & \mathcal{F}^{top}_{\infty}(\Lambda(n_1, n_2)) \arrow[d, "S^{\Lambda(l_1,l_2)}_{\infty}"] \\
\mathcal{F}^{top}_{\infty}(\Lambda(n'_1, n'_2)) \arrow[r, "S^{\Lambda(l_1,l_2)}_{\infty}"] & \mathcal{F}^{top}_{\infty}(\Lambda(l_1, l_1)).
\end{tikzcd}$$

Proof. There is a morphism of diagrams

$$\begin{tikzcd}
 & \mathcal{F}^{top}_{\infty}(\Lambda(n_1,n_2)) \arrow[r, bend left, "T^{\Lambda(m_1,m_2)}_{\infty}"] \arrow[d, "S^{\Lambda(l_1,l_2)}_{\infty}"'] & \mathcal{F}^{top}_{\infty}(\Lambda(m_1,m_2)) \arrow[d, "Id"] & \\
\mathcal{F}^{top}_{\infty}(\Lambda(n_1',n_2')) \arrow[r, "S^{\Lambda(l_1,l_2)}_{\infty}"] \arrow[rrr, bend right, "T^{\Lambda(m_1,m_2)}_{\infty}"'] & \mathcal{F}^{top}_{\infty}(\Lambda(l_1,l_2)) \arrow[r, bend left, "T^{\Lambda(m_1,m_2)}_{\infty}"] & \mathcal{F}^{top}_{\infty}(\Lambda(m_1,m_2)) & \mathcal{F}^{top}_{\infty}(\Lambda(m_1,m_2)) \arrow[l, "Id"']
\end{tikzcd}$$

where all the curved horizontal arrows are fully-faithful embeddings. Thus the fiber-product of the diagram on the left embeds fully-faithfully in the fiber product of the diagram on the right, which is given $\mathcal{F}^{top}_{\infty}(\Lambda(m_1,m_2))$. In order to prove that this embedding is an equivalence, we need to show that a set of generators of $\mathcal{F}^{top}_{\infty}(\Lambda(m_1,m_2))$ lies in the image. This can be done easily by expressing $\mathcal{F}^{top}_{\infty}(\Lambda(m_1,m_2))$ as a category of quiver representations as in Lemma 6.3.

We also give a different proof based on mirror symmetry in the special case

$$\Lambda(n_1,0) \subset \Lambda(n_1,n_2) \supset \Lambda(0,n_2), \quad \Lambda(n_1,0) \cap \Lambda(0,n_2) = \Lambda(0,0)$$

since this clarifies the connection between gluing along closed subskeleta and Zariski descent. Let $\mathbb{P}^1(n_1,n_2)$ be the projective line with two stacky points at 0 and ∞ having isotropy isomorphic to the groups of roots of unity μ_{n_1} and μ_{n_2}. More formally, $\mathbb{P}^1(n_1,n_2)$ is the push-out of the following diagram in the category of DM stacks,

$$\begin{tikzcd}
 & \mathbb{G}_m \arrow[dl] \arrow[dr] & \\
{[\mathbb{A}^1/\mu_{n_1}]} & & {[\mathbb{A}^1/\mu_{n_2}],}
\end{tikzcd}$$

where $[\mathbb{A}^1/\mu_{n_1}]$ and $[\mathbb{A}^2/\mu_{n_2}]$ are the quotient stacks of $\mathbb{A}^1$ under the canonical action of μ_{n_1} and μ_{n_2}. Zariski descent implies that the diagram

$$\begin{tikzcd}
\mathcal{QC}oh^{(2)}(\mathbb{P}^1(n_1,n_2)) \arrow[r] \arrow[d] & \mathcal{QC}oh^{(2)}([\mathbb{A}^1/\mu_{n_1}]) \arrow[d] \\
\mathcal{QC}oh^{(2)}([\mathbb{A}^1/\mu_{n_2}]) \arrow[r] & \mathcal{QC}oh^{(2)}(\mathbb{G}_m),
\end{tikzcd} \tag{8}$$

where all the arrows are pullbacks, is a fiber product. It follows from [STZ] and [DK] that diagram (8) is in fact equivalent to the diagram in the statement of the lemma. More precisely, there are commutative diagrams

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(\Lambda(n_1,n_2)) \arrow[r, "S^{\Lambda(n_1,0)}_{\infty}"] \arrow[d, "\simeq"'] & \mathcal{F}^{top}_{\infty}(\Lambda(n_1,0)) \arrow[d, "\simeq"] & \mathcal{F}^{top}_{\infty}(\Lambda(n_1,n_2)) \arrow[r, "S^{\Lambda(n_2,0)}_{\infty}"] \arrow[d, "\simeq"'] & \mathcal{F}^{top}_{\infty}(\Lambda(0,n_2)) \arrow[d, "\simeq"] \\
\mathcal{QC}oh^{(2)}(\mathbb{P}^1(n_1,n_2)) \arrow[r] & \mathcal{QC}oh^{(2)}([\mathbb{A}^1/\mu_{n_1}]), & \mathcal{QC}oh^{(2)}(\mathbb{P}^1(n_1,n_2)) \arrow[r] & \mathcal{QC}oh^{(2)}([\mathbb{A}^1/\mu_{n_2}]),
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(\Lambda(n_1,0)) \arrow[r, "S_{\infty}^{\Lambda(0,0)}"] \arrow[d, "\simeq"'] & \mathcal{F}^{top}_{\infty}(\Lambda(0,0)) \arrow[d, "\simeq"] & \mathcal{F}^{top}_{\infty}(\Lambda(0,n_2)) \arrow[r, "S_{\infty}^{\Lambda(0,0)}"] \arrow[d, "\simeq"'] & \mathcal{F}^{top}_{\infty}(\Lambda(0,0)) \arrow[d, "\simeq"] \\
\mathcal{Q}Coh^{(2)}([\mathbb{A}^1/\mu_{n_1}]) \arrow[r] & \mathcal{Q}Coh^{(2)}(\mathbb{G}_m), & \mathcal{Q}Coh^{(2)}([\mathbb{A}^1/\mu_{n_2}]) \arrow[r] & \mathcal{Q}Coh^{(2)}(\mathbb{G}_m).
\end{tikzcd}$$

such that all vertical arrows are equivalences. Since diagram (8) is a fiber product we conclude that also the diagram in the statement of the lemma is a fiber product. ☐

The following is the main result of this section. In order to avoid cluttering the diagrams, we denote restrictions and exotic restrictions simply R_∞ and S_∞.

Theorem 6.5. *Let X be a ribbon graph. Let Z_1 and Z_2 be good closed subgraphs such that:*

- $Z_1 \cup Z_2 = X$
- *The underlying topological space of $Z_{1,2} := Z_1 \cap Z_2$ is disjoint union of circles.*

Then the commutative diagram

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(X) \arrow[r, "S_\infty"] \arrow[d, "S_\infty"'] & \mathcal{F}^{top}_{\infty}(Z_1) \arrow[d, "S_\infty"] \\
\mathcal{F}^{top}_{\infty}(Z_2) \arrow[r, "S_\infty"] & \mathcal{F}^{top}_{\infty}(Z_{1,2}).
\end{tikzcd}$$

is a fiber product in $\widehat{dgCat^{(2)}}$.

We will assume for simplicity that $Z_{1,2}$ has only one connected component: the general case is proved in the same way. Proving Theorem 6.5 will require some preparation.

Let X be a ribbon graph, and let Z be a closed subgraph. It is useful to consider a combinatorial analogue of a tubular neighborhood of Z inside X, which we denote $N_Z X$: the graph $N_Z X$ is given by Z plus additional open edges for each edge in X that does not lie in Z, but is incident to a vertex in Z. Here is a formal definition. We subdivide all the edges of X which do not lie in Z, but whose endpoints lie in Z. We denote the resulting graph again X: from now on, every time we will consider the object $N_Z X$, we will assume implicitly that the edges of X are sufficiently subdivided. Let $\overline{Z}^c$ be the maximal (non-necessarily good) closed subgraph of X such that

$$Z \cap \overline{Z}^c = \varnothing.$$

We denote $N_Z X$ the open subgraph

$$X - \overline{Z}^c \subset X$$

Now let X, Z_1 and Z_2 be as in Theorem 6.5, and assume that $Z_{1,2}$ has only one connected component. Note that $N_{Z_{1,2}}X$, $N_{Z_{1,2}}Z_1$ and $N_{Z_{1,2}}Z_2$ are all wheel-type graphs. We make the following notations:

- $U_1 = Z_1 \cup N_{Z_{1,2}}X$. The graph U_1 is an open subgraph of X and Z_1 is a good closed subgraph of U_1
- $U_2 = Z_2 \cup N_{Z_{1,2}}X$. The graph U_2 is an open subgraph of X and Z_2 is a good closed subgraph of U_2

- $U_{1,2} = U_1 \cap U_2$
- $U_1^o = Z_1 - Z_{1,2}$, and $U_2^o = Z_2 - Z_{1,2}$, where the superscript o stands for *open*. The graphs U_1^o and U_2^o are open subgraphs of X
- $U_1^e = U_{1,2} \cap U_1^o$ and $U_2^e = U_{1,2} \cap U_2^o$, where the superscript e stands for *edges*

Note that $U_{1,2}$ is equal to $N_{Z_{1,2}}X \cong \Lambda(m_1, m_2)$, and that the embeddings

$$U_1^e \subset U_{1,2}, \quad U_2^e \subset U_{1,2}$$

are isomorphic to the embeddings of the spokes

$$E(+) \subset \Lambda(m_1, m_2), \quad E(-) \subset \Lambda(m_1, m_2).$$

Also, we have identifications $N_{Z_{1,2}}Z_1 = Z_1 \cap U_{12}$, and $N_{Z_{1,2}}Z_2 = Z_2 \cap U_{12}$.

The key ingredient in the proof of Theorem 6.5 is the following lemma.

Lemma 6.6. *All the interior squares in the commutative diagram*

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(X) \arrow[r, "R_\infty"] \arrow[d, "R_\infty"'] & \mathcal{F}^{top}_{\infty}(U_1) \arrow[r, "S_\infty"] \arrow[d, "R_\infty"'] & \mathcal{F}^{top}_{\infty}(Z_1) \arrow[d, "R_\infty"] \\
\mathcal{F}^{top}_{\infty}(U_2) \arrow[r, "R_\infty"] \arrow[d, "S_\infty"'] & \mathcal{F}^{top}_{\infty}(U_{1,2}) \arrow[r, "S_\infty"] \arrow[d, "S_\infty"'] & \mathcal{F}^{top}_{\infty}(Z_1 \cap U_{1,2}) \arrow[d, "S_\infty"] \\
\mathcal{F}^{top}_{\infty}(Z_2) \arrow[r, "R_\infty"] & \mathcal{F}^{top}_{\infty}(Z_2 \cap U_{1,2}) \arrow[r, "S_\infty"] & \mathcal{F}^{top}_{\infty}(Z_{1,2})
\end{tikzcd}$$

are fiber products.

Proof. Number clockwise the interior squares from one to four, starting with the top left one. Square 1 is a fiber product by Proposition 5.7. Square 3 is a fiber product by Lemma 6.4. Up to swapping U_1 with U_2, squares 2 and 4 are identical. So it is enough to prove that square 2 is a fiber product. The proof consists of three steps.

Step one: We express all the vertices of square 2 as fiber products. We start with the top vertices. Each of the following two diagrams

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(U_1) \arrow[r, "R_\infty"] \arrow[d, "R_\infty"'] & \mathcal{F}^{top}_{\infty}(U_{1,2}) \arrow[d, "R_\infty"] & \mathcal{F}^{top}_{\infty}(Z_1) \arrow[r, "R_\infty"] \arrow[d, "R_\infty"'] & \mathcal{F}^{top}_{\infty}(Z_1 \cap U_{1,2}) \arrow[d, "R_\infty"] \\
\mathcal{F}^{top}_{\infty}(U_1^o) \arrow[r, "R_\infty"] & \mathcal{F}^{top}_{\infty}(U_1^e) & \mathcal{F}^{top}_{\infty}(U_1^o) \arrow[r, "R_\infty"] & \mathcal{F}^{top}_{\infty}(U_1^e).
\end{tikzcd} \tag{9}$$

is a fiber product in $\widehat{dgCat^{(2)}}$ by Proposition 5.7. Let us consider the bottom vertices next. The diagrams

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(U_{1,2}) \arrow[r, "R_\infty"] \arrow[d, "R_\infty"'] & \mathcal{F}^{top}_{\infty}(U_{1,2}) \arrow[d, "R_\infty"] & \mathcal{F}^{top}_{\infty}(Z_1 \cap U_{1,2}) \arrow[r, "R_\infty"] \arrow[d, "R_\infty"'] & \mathcal{F}^{top}_{\infty}(Z_1 \cap U_{1,2}) \arrow[d, "R_\infty"] \\
\mathcal{F}^{top}_{\infty}(U_1^e) \arrow[r, "R_\infty"] & \mathcal{F}^{top}_{\infty}(U_1^e) & \mathcal{F}^{top}_{\infty}(U_1^e) \arrow[r, "R_\infty"] & \mathcal{F}^{top}_{\infty}(U_1^e).
\end{tikzcd} \tag{10}$$

are trivially fiber products in $\widehat{dgCat}^{(2)}$ since the horizontal arrows are identities, and any two parallel arrows are equal to each other.

Step two: The arrows in square 2 can be written in terms of morphisms between the fiber product diagrams constructed in step one. Let us focus, for instance, on the bottom horizontal map in square 2

$$\mathcal{F}^{top}_\infty(U_{1,2}) \overset{S_\infty}{\to} \mathcal{F}^{top}_\infty(Z_1 \cap U_{1,2}).$$

This map, which is indicated in the diagram below by a dashed arrow, is induced by the map of diagrams in $\widehat{dgCat}^{(2)}$ given by the three arrows in the middle: S_∞, Id and Id,

$$\begin{tikzcd}
 & \mathcal{F}^{top}_\infty(U_{1,2}) \arrow[r, "S_\infty"] \arrow[d] & \mathcal{F}^{top}_\infty(Z_1 \cap U_{1,2}) \arrow[d] & \\
\mathcal{F}^{top}_\infty(U_{1,2}) \arrow[ur] \arrow[dr] \arrow[rrr, dashed, bend left=15] & \mathcal{F}^{top}_\infty(U_1^e) \arrow[r, "Id"] & \mathcal{F}^{top}_\infty(U_1^e) & \mathcal{F}^{top}_\infty(Z_1 \cap U_{1,2}) \arrow[ul] \arrow[dl] \\
 & \mathcal{F}^{top}_\infty(U_1^e) \arrow[r, "Id"] \arrow[u] & \mathcal{F}^{top}_\infty(U_1^e) \arrow[u] &
\end{tikzcd}$$

A similar reasoning holds also for the other arrows in square 2.

Step three: We complete the proof by using the fact that limits commute with limits. We have to show that square 2, which we reproduce as diagram (11), is a fiber product.

$$\begin{tikzcd}
\mathcal{F}^{top}_\infty(U_1) \arrow[r, "R_\infty"] \arrow[d, "S_\infty"'] & \mathcal{F}^{top}_\infty(Z_1) \arrow[d, "R_\infty"] \\
\mathcal{F}^{top}_\infty(U_{1,2}) \arrow[r, "R_\infty"] & \mathcal{F}^{top}_\infty(Z_1 \cap U_{1,2})
\end{tikzcd} \tag{11}$$

We can commute (11) and the fiber products constructed in step one past each other: thus, in order to show that (11) is a fiber product, we can prove instead that the following are fiber products

$$\begin{tikzcd}
\mathcal{F}^{top}_\infty(U_{1,2}) \arrow[r, "S_\infty"] \arrow[d, "Id"'] & \mathcal{F}^{top}_\infty(Z_1 \cap U_{1,2}) \arrow[d, "Id"] & \mathcal{F}^{top}_\infty(U_1^o) \arrow[r, "Id"] \arrow[d, "R_\infty"'] & \mathcal{F}^{top}_\infty(U_1^o) \arrow[d, "R_\infty"] & \mathcal{F}^{top}_\infty(U_1^e) \arrow[r, "Id"] \arrow[d, "Id"'] & \mathcal{F}^{top}_\infty(U_1^e) \arrow[d, "Id"] \\
\mathcal{F}^{top}_\infty(U_{1,2}) \arrow[r, "S_\infty"] & \mathcal{F}^{top}_\infty(Z_1 \cap U_{1,2}) & \mathcal{F}^{top}_\infty(U_1^e) \arrow[r, "Id"] & \mathcal{F}^{top}_\infty(U_1^e) & \mathcal{F}^{top}_\infty(U_1^e) \arrow[r, "Id"] & \mathcal{F}^{top}_\infty(U_1^e)
\end{tikzcd} \tag{12}$$

These diagrams have the property that the horizontal arrows are identities, and any two parallel arrows are equal to each other: so they are fiber products. This concludes the proof. □

The proof of Proposition 6.5. Note first that the diagram from the statement of Proposition 6.5 is the exterior square of the diagram from Lemma 6.6. Indeed by Proposition 5.14

$$S_\infty \simeq S_\infty \circ R_\infty.$$

By general properties of fiber products, since all the interior squares are fiber products, the exterior one is a fiber product as well. This concludes the proof. □

Remark 6.7. Although Theorem 6.5 is sufficient for our purposes, we expect that gluing formulas under closed covers hold much more generally. The importance of this kind of gluing formulas lies in the fact that they are powerful computational tools, and that they often correspond via mirror symmetry to Zariski descent statement for quasi-coherent sheaves and matrix factorizations (see, for instance, the proof of Lemma 6.4). We will return to the problem of developing a comprehensive formalism of gluing formulas along closed subskeleta for $\mathcal{F}^{top}$, in dimension two and higher, in future work.

7. Tropical and Surface topology

7.1. **Surface topology.** This section contains some remarks on surface topology that will be useful in later constructions.

Denote by $\Sigma_{g,n}$ an oriented surface of genus g with n punctures. Since the topology of these surfaces enters our discussion in a relatively coarse way, we will often draw the punctures as if they were boundaries, but strictly speaking $\Sigma_{g,n}$ is a noncompact (if $n > 0$) manifold without boundary. The surface $\Sigma_{g,n}$ has n *ends* corresponding to the punctures.[4]

If Σ_1 and Σ_2 are two oriented punctured surfaces, we may form a new surface by the well-known *end connect sum* operation.

Definition 7.1. Choose a puncture p_1 on Σ_1 and a puncture p_2 on Σ_2. Identify a neighborhood of p_1 with $S^1 \times (-1, -1/2)$ and a neighborhood of p_2 with $S^1 \times (1/2, 1)$, and replace the union of these neighborhoods by a single punctured cylinder $S^1 \times (-1, 1) \setminus (1, 0)$. The result $\Sigma_1 \#_{p_1,p_2} \Sigma_2$ is called the *end connect sum of* Σ_1 *and* Σ_2 *at the punctures* p_1 *and* p_2.

The end connect sum can also be described as attaching a one-handle to $\Sigma_1 \coprod \Sigma_2$. If alternatively we think in terms of bordered surfaces, the operation consists of adding a strip connecting two boundary components. If Σ_i has genus g_i and n_i punctures ($i = 1, 2$), then $\Sigma \#_{p_1,p_2} \Sigma_2$ has genus $g_1 + g_2$ and $n_1 + n_2 - 1$ punctures.

The effect of end connect sum on skeleta is straightforward.

Lemma 7.2. *Let X_i be a skeleton for Σ_i ($i = 1, 2$). Produce from X_i a ribbon graph with a noncompact edge connecting X_i to the puncture p_i; call the result X_i'. Then a skeleton for $\Sigma_1 \#_{p_1,p_2} \Sigma_2$ is obtained by connecting the noncompact edges of X_1' and X_2' inside the attaching region.*

Example 7.3. We can decompose $\Sigma_{g,n}$ ($n > 0$) into an iterated end connect sum of $\Sigma_{1,1}$ and $\Sigma_{0,2}$. Indeed, taking end connect sum of g copies of $\Sigma_{1,1}$ (always summing at the unique punctures) yields a surface of type $\Sigma_{g,1}$. Taking end connect sum of $n - 1$ copies of $\Sigma_{0,2}$ (summing at a single puncture of each) yields a surface of type $\Sigma_{0,n}$. Then end connect summing $\Sigma_{g,1}$ and $\Sigma_{0,n}$ yields $\Sigma_{g,n}$. By choosing skeleta for $\Sigma_{1,1}$ (consisting, say, of two loops on the torus) and for $\Sigma_{0,2}$ (say a single circle), we thus obtain a skeleton for $\Sigma_{g,n}$. This is pictured in Figure 2.

In this paper, we are interested with skeleta with a certain shape near the punctures.

---

[4] An *end* of a topological space X is a function ϵ from the set of compact subsets of $K \subset X$ to subsets of X, such that $\epsilon(K)$ is a connected component of $X \setminus K$, and such that if $K_1 \subset K_2$, then $\epsilon(K_2) \subset \epsilon(K_1)$. Thus ends are intrinsic to the space X, and make sense without reference to a compactification.

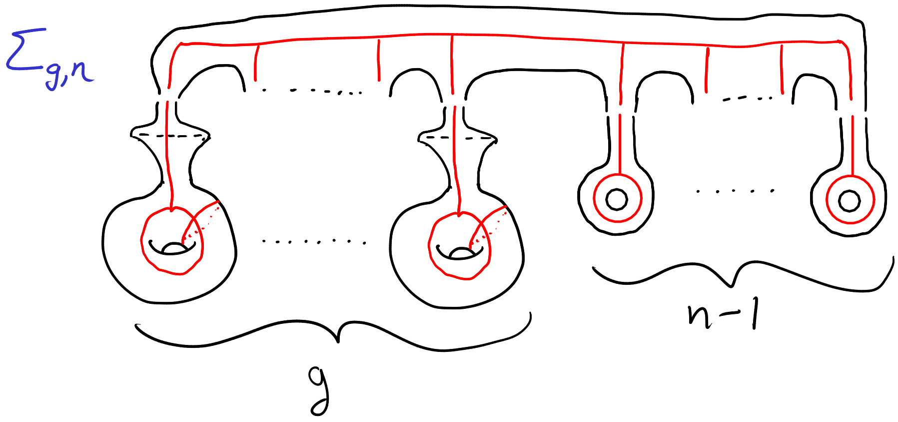

FIGURE 2. Decomposition of $\Sigma_{g,n}$ into end connect sum of g copies of $\Sigma_{1,1}$ and $n-1$ copies of $\Sigma_{0,2}$

Definition 7.4. Let Σ be a punctured surface, p a puncture of Σ, and $X \subset \Sigma$ a skeleton for Σ. The component of $\Sigma \setminus X$ containing the puncture p is topologically a punctured disk, and its boundary is a subgraph of X. We say that X *has a cycle at the puncture* p if this subgraph is a cycle.

If p_1 and p_2 are distinct punctures, we say that X has *disjoint cycles* at p_1 and p_2 if it has cycles at p_1 and p_2, and these cycles are disjoint in X.

Model the pair of pants as $\mathbb{C} - \{-2, 2\}$. If x and y are in $\mathbb{C}$, and ϵ is a positive real number, we denote $S(x, \epsilon) \subset \mathbb{C}$ the circle of center x and radius ϵ, and $I(x, y) \subset \mathbb{C}$ the straight segment joining x and y. We call Θ *graph* the skeleton of the pair of pants given by

$$S(0,3) \cup I(-3i, 3i).$$

We call *dumbell graph* the skeleton given by

$$S(-2,1) \cup I(-1,1) \cup S(2,1).$$

The Θ graph has a cycle at each of the three punctures, whereas the dumbbell graph has a cycle at only two: for the third puncture, the boundary of the corresponding component of $\Sigma \setminus X$ consists of the entire skeleton. On the other hand, in the Θ graph the cycles for any two punctures are not disjoint, whereas in the dumbbell graph they are disjoint. These graphs are shown in Figure 3.

Lemma 7.5. *$\Sigma_{g,n}$ admits a skeleton that has a cycle at every puncture but one.*

Proof. This is furnished by Example 7.3. ☐

In fact, whenever X is a skeleton with a cycle at a particular puncture, Σ and X can be decomposed into an end connect sum in a manner similar to that of Example 7.3.

Lemma 7.6. *Let X be a ribbon graph for Σ that has a cycle at the puncture p. Suppose that r other edges are incident to the cycle at p. The Σ can be decomposed into an end connect*

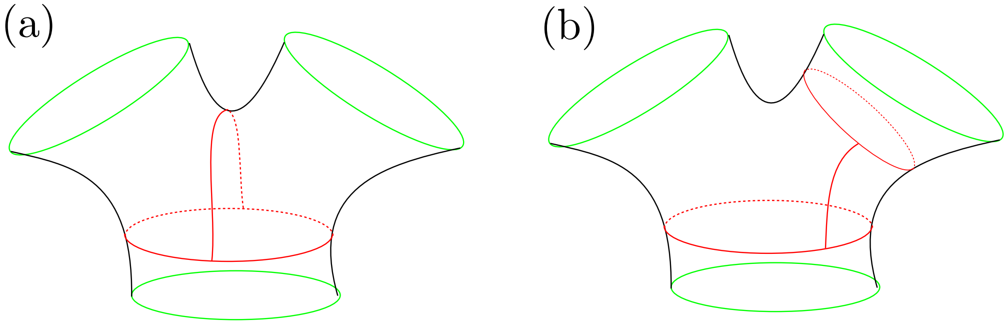

FIGURE 3. Two skeleta for the pair of pants. (a) Θ graph. (b) Dumbbell graph.

sum of Σ' *and* Σ''*, where* $\Sigma'' \cong \Sigma_{0,2}$*, and* Σ' *has one fewer puncture than* Σ*, and* X *can be decomposed into the sum of* X' *and* X''*, where* X' *and* X'' *are ribbon graphs embedded in* Σ' *and* Σ'' *respectively, each with* r *noncompact edges approaching the punctures where the connect sum is taken, and* X *is obtained by connecting the noncompact edges of* X' *to those of* X''*. See Figure 4.*

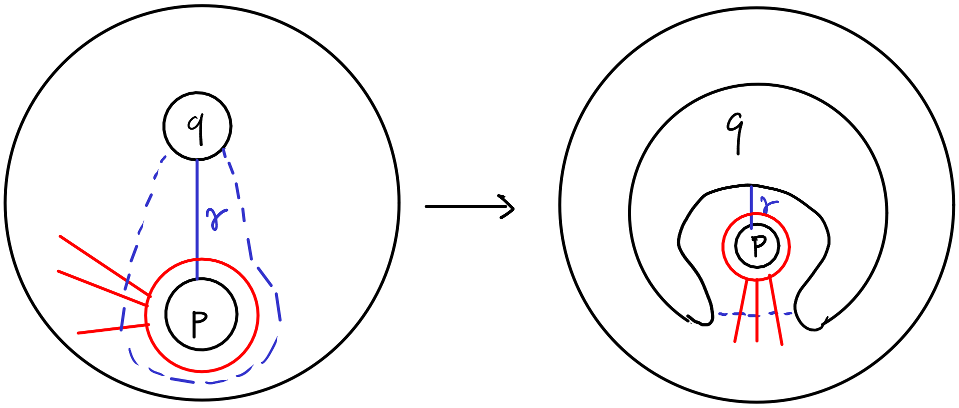

FIGURE 4. Decomposition of Σ into an end connect sum, depending on a choice of path γ between two punctures. The dotted line is for visual reference.

Proof. The idea is to deform our picture of Σ so that the cycle at p is pulled out at another puncture q of Σ. To do this, what is needed is a path γ in Σ from p to q that does not cross any other points of the skeleton X. But this is always possible, since every component of $\Sigma \setminus X$ is homeomorphic to a punctured disk. ☐

We remark that the proof shows that if r edges are incident to the cycle at p, then there are essentially r choices for how to decompose Σ and X as in the lemma.

Lemma 7.7. (1) *Let* X_1 *and* X_2 *be two ribbon graphs for* Σ *that both have cycles at the puncture* p*. Then it is possible to connect* X_1 *to* X_2 *by a sequence of contractions and expansions so that every intermediate graph also has a cycle at* p*.*

(2) *Let Σ be a surface with at least three punctures. Let p be a puncture of Σ, and let X be a skeleton for Σ that has a cycle at p. Let p' be another puncture of X. It is possible to modify X to X' so that X' also has cycles at p and p', and so that every intermediate graph also has a cycle at p.*

Proof. (1) First, if there is more than one edge incident to the cycle at p in X_1 or X_2, we can apply contractions to gather together all of these edges into a single vertex of valence $r+2$, and then apply a single expansion to ensure that in both X_1 and X_2 only a single edge is incident to the cycle at p. None of these moves destroy the cycle at p in X_1 or X_2.

Let $p' \neq p$ be another puncture. Choose a path γ from p to p'. As in Lemma 7.6, we may assume that γ only crosses X_1 at the cycle at p. Once this choice is made, we cannot assume the same holds true for X_2, so γ will cross X_2 at some number of edges not contained in the cycle at p; let k be this number. Now we apply the idea of stretching the surface from 7.6, using the chosen path γ. This decomposes Σ into an end connect sum of Σ' and Σ'', where Σ'' is has genus 0 and 2 punctures in such a way that the cycle at p ends up in the Σ'' factor. See Figure 5(a).

Now we consider how X_1 and X_2 look with respect to this decomposition. Since the path γ only intersects X_1 at the cycle at p, X_1 decomposes just as in Lemma 7.6. On the other hand, X_2 is as shown in Figure 5(a). The part of X_2 that ends up in Σ'' consists of a cycle at p together with k parallel arcs. This is connected to the rest of X_2 via $2k+1$ noncompact edges.

The next step is to apply moves to X_2 that move the k arcs out of Σ'' and into Σ'. Observe that the space between two neighboring arcs corresponds, in the summed surface Σ, to a component of $\Sigma \setminus X_2$, which is a punctured disk. Start with the outermost arc, call it a. Let D denote the punctured disk corresponding to the region just inside a, so D is a punctured disk. The arc a ends at two vertices in Σ'. By a sequence of contractions and expansions, we may move one of the ends along the boundary of D, through Σ'', and back into Σ'. We can also follow the disk D throughout this process. (Depending on how it is done, the puncture of D may also move through Σ''.) This is depicted in Figure 5(b). Since none of these moves destroy the cycle at p, this reduces us to the situation where $k=0$.

In the case $k=0$, we have decompositions of X_1 and X_2 into end connect sums of X_1' and X_2' in Σ', each having a single noncompact edge, and X'' in Σ'' consisting of a single cycle with a single noncomapact edge. Now we apply the fact that any two ribbon graphs for Σ' with a single noncompact edge asymptotic to a puncture can be connected by a sequence of moves, by a result of Harer [DK, Proposition 3.3.9]. Evidently, such moves do not destroy the cycle at the puncture in Σ'', so we are done.

(2) Since the surface Σ has at least three punctures, there is a ribbon graph X' that has cycles at both p and p'. Now apply the first part of the lemma.

□

7.2. **Tropical topology.** Since our strategy is to prove HMS inductively by gluing together pairs of pants, and the gluings are controlled by a balanced tropical graph $G_{\mathcal{T}}$ associated

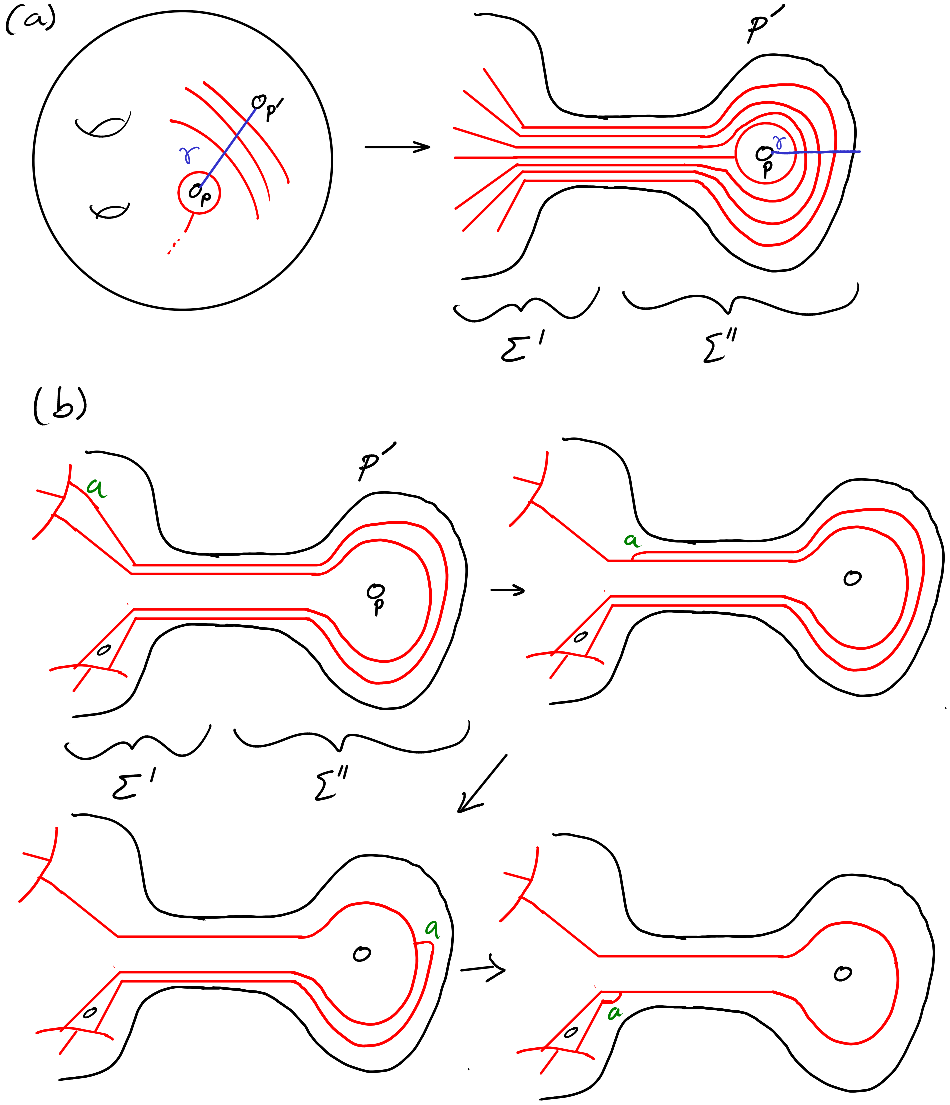

FIGURE 5. (a) Decomposition of surface into end connect sum, and corresponding decomposition of ribbon graph. (b) Moving an arc through Σ''. The point marked a is the end of the arc that is being moved.

to the given toric Calabi-Yau Landau-Ginzburg model $(X_{\mathcal{T}}, W_{\mathcal{T}})$, we collect here some elementary remarks about the topology of such graphs that will be useful. The main point is to keep track of the non-compact edges of G, since these are edges where we *never* need to glue in our induction; we also point out that G can be built up in such a way that we never need to glue along all the edges incident to a single vertex.

Let G be trivalent graph with both finite and infinite edges. For each edge e, we have an *orientation line* $\det(e)$ that is the $\mathbb{Z}$-module generated by the two orientations of e modulo the relation that their sum is zero. A *(planar integral) momentum vector* p_e on the edge e is a linear map $p_e : \det(e) \to \mathbb{Z}^2$.

Definition 7.8. A pair $(G, \{p_e\}_{e\in\mathrm{Edge}(G)})$ consiting of a graph and a set of momenta is a *balanced tropical graph* if momentum is conserved at each vertex. Namely, for each vertex v of G,

$$\sum_e p_e(\text{inward orientation}) = 0 \tag{13}$$

where the sum is over all edges e incident to v, and the inward orientation is the one pointing toward v.

Such a graph is additionally called *nondegenerate* if the values of the momenta at each vertex span $\mathbb{Z}^2$, or equivalently if not all momenta at a given vertex are proportional.

Definition 7.9. A *planar immersion* of $(G, \{p_e\}_{e\in\mathrm{Edge}(G)})$ is a continuous map $i : G \to \mathbb{R}^2$, such that derivative of i along an edge e in the direction o is positively proportional to the momentum $p_e(o)$. Note that we do not require i to be proper on infinite edges.

From now on we will consider nondegenerate balanced tropical graphs $(G, \{p_e\}_{e\in\mathrm{Edge}(G)})$ with planar immersion i. Planar immersions of balanced tropical graphs are in some sense "harmonic," so it is not surprising that they satisfy a version of the maximum principle:

Lemma 7.10. *G has at least two infinite edges.*

Proof. Let $i : G \to \mathbb{R}^2$ be a planar immersion, and let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the orthgonal projection onto any given line of irrational slope. Then consider the function $\pi \circ i : G \to \mathbb{R}$. Nondegeneracy implies that $p_e \neq 0$ for any e, so $\pi \circ i$ is not constant on any edge.

We claim that $\pi \circ i$ does not achieve its supremum. For if it did, this would necessarily occur at a vertex, as $\pi \circ i$ is linear and nonconstant on all edges. At the vertex, the images under i of all incident edges lie in the same half-plane determined by the linear function π. This is not compatible with the balancing condition, since three non-zero vectors in the same half-plane cannot sum to zero.

The same reasoning applied to $-\pi \circ i$ shows that $\pi \circ i$ does not achieve its infimum. Therefore there must be two infinite edges on which the supremum and infinimum are approached but not obtained. ☐

Now let e_0 be an infinite edge of G; it is incident to a vertex v_0, and there are three possibilities for the local structure of G at v_0:

1. v_0 is incident to one infinite edge, namely e_0.
2. v_0 is incident to two infinite edges, namely e_0 and one other e_1.
3. v_0 is incident to three infinite edges. Then v_0 and these three edges comprise a connected component of G.

See Figure 6.

Lemma 7.11. *In case 1, let G' be the graph obtained from G by deleting e_0 and v_0. In case 2, let G' be the graph obtained from G by deleting e_0, e_1 and v_0. Then G' has an infinite edge not originally incident to v_0 (in G).*

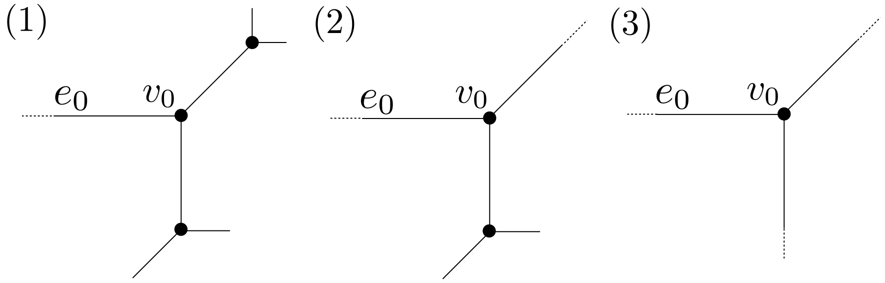

Figure 6. Cases at a vertex.

Proof. In case 2, this follows from Lemma 7.10, as G' must have two infinite edges, and only one is incident to v_0.

In case 1, let e_1 and e_2 be the other edges incident to v_0; these become infinite edges in G'. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be orthgonal projection onto an irrational line chosen so that both $i(e_1)$ and $i(e_2)$ lie in the half-plane defined by the inequality $\pi(x) \geq \pi(i(v_0))$. The argument from the proof of Lemma 7.10 shows that $\pi \circ i$ approaches its supremum along some infinite edge. This edge cannot be e_1 or e_2, as $\pi \circ i$ is *decreasing* in the noncompact direction on these edges. ☐

Lemma 7.12. *Given $i : G \to \mathbb{R}^2$ a planer immersion, there exists a sequence $i_j : G_j \to \mathbb{R}^2$, $j = 1, \dots, N$ with the following properties.*

1. *$i_j : G_j \to \mathbb{R}^2$ is planer immersion of the tropical graph G_j,*
2. *$i_N : G_N \to \mathbb{R}^2$ equals $i : G \to \mathbb{R}^2$.*
3. *There is a continuous embedding $G_j \to G_{j+1}$ such that $i_j = i_{j+1}|G_j$, and such that G_{j+1} is obtained from G_j by gluing a single trivalent vertex to G_j along either one or two of the noncompact edges of G_j, and also extending some other noncompact edges of G_j.*

Proof. Begin with $i : G \to \mathbb{R}^2$, and once again choose a projection $\pi : \mathbb{R}^2 \to \mathbb{R}$. If π is chosen generically, each fiber of $\pi \circ i$ will contain at most one vertex of G. Let the values of $\pi \circ i$ on the vertices be $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$. Then take $G_i = (\pi \circ i)^{-1}(-\infty, \lambda_i)$. ☐

8. The induction

This section contains the main induction that proves HMS.

For any oriented punctured surface Σ equipped with a skeleton X, we associate the topological Fukaya category $\mathcal{F}^{top}_{\infty}(X)$. For each puncture p of Σ, there is a restriction functor $R_p : \mathcal{F}^{top}_{\infty}(X) \to \mathcal{F}^{top}_{\infty}(S^1)$. Here S^1 denotes a ribbon graph consisting of a single cycle; it is a skeleton for the cylinder. If the graph X contains a cycle corresponding to the puncture p, then R is defined directly as a closed restriction restriction functor. If not, then R is defined by first choosing another skeleton X' that does have cycle corresponding to the puncture p. There is a canonical equivalence between $\Phi_{X,X'} : \mathcal{F}^{top}_{\infty}(X) \to \mathcal{F}^{top}_{\infty}(X')$ and a closed restriction functor $R : \mathcal{F}^{top}_{\infty}(X') \to \mathcal{F}^{top}_{\infty}(S^1)$ corresponding to p; composing these gives the desired restriction functor. We first show that this functor does not depend on the choice of skeleton used to define it.

Lemma 8.1. *Let X_1 and X_2 be two skeleta for Σ that both have cycles corresponding to the puncture p. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(X_1) \arrow[r, "\Phi"] \arrow[dr, "R"'] & \mathcal{F}^{top}_{\infty}(X_2) \arrow[d, "R"] \\
 & \mathcal{F}^{top}_{\infty}(S^1)
\end{tikzcd} \tag{14}$$

where Φ denotes the canonical equivalence, and R denotes closed restriction maps.

Proof. This is an application of Lemma 7.7. Since we an arrange that the contractions and expansions that implement Φ do not destroy the cycle at p, at every step the desired commutative diagram both makes sense and holds true. ☐

By Definition 3.8 for any nondegenerate balanced graph with planar immersion G, we have a matrix-factorization-type category $\mathcal{B}(G)$. For each external edge of G, there is a restriction functor $\mathcal{B}(G) \to \mathcal{B}(E)$, where E is the graph consisting of a single bi-infinite edge. The graph G determines a punctured Riemann surface $\Sigma(G)$ in a way that generalizes the familiar correspondence between an algebraic curve and its tropicalization: namely, the genus of $\Sigma(G)$ is the number of relatively compact connected components in $\mathbb{R}^2 - G$, and the number of punctures is given by the number of infinite edges of G.

Now we come to the main result, that category $\mathcal{F}^{top}_{\infty}(X)$ is equivalent to the category $\mathcal{B}(G)$ (see Definition 3.8), where X is a skeleton for $\Sigma(G)$. Since our method involves successively gluing pairs of pants inductively, we must include in the induction a statement on the restriction maps at the punctures.

Theorem 8.2. *If X is a skeleton for $\Sigma(G)$, then there is an equivalence of categories $\Psi : \mathcal{F}^{top}_{\infty}(X) \to \mathcal{B}(G)$ with the property that for each infinite edge e of G, and corresponding puncture $p(e)$, there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(X) \arrow[r, "\Psi"] \arrow[d, "R_{p(e)}"'] & \mathcal{B}(G) \arrow[d, "R"] \\
\mathcal{F}^{top}_{\infty}(S^1) \arrow[r, "\Psi"] & \mathcal{B}(e)
\end{tikzcd} \tag{15}$$

where the vertical arrows are restriction functors.

Proof. We may regard the graph G as being constructed from a collection of trivalent vertices by gluing infinite edges to each other. By Lemma 7.12, there is a collection of graphs G_i, $i = 1, \dots, N$ such that $G_N = G$, and G_{i+1} is obtained from G_i by gluing a single trivalent vertex to either one or two infinite edges of G_i (but not at all three edges simultaneously).

We shall prove the assertions in the theorem by induction on i. In the base case $i = 1$, we are simply considering the pair of pants, for which the result is known.

For the induction step, the induction hypothesis is the statement of the theorem for G_i. In passing from G_i to G_{i+1}, we attach a trivalent vertex T; correspondingly, $\Sigma(G_{i+1})$ is obtained from $\Sigma(G_i)$ by attaching a pair of pants $\Sigma(T)$. Now there are two cases, depending on whether the gluing involves one edge or two.

Case of one edge: Let $e \in G_i$ and $e' \in T$ denote the edges that are being glued. Then $\Sigma(G_i)$ has a puncture $p(e)$ and $\Sigma(T)$ has a puncture $p(e')$. We may choose skeleta X for $\Sigma(G_i)$ and Y for $\Sigma(T)$ such that X has a cycle at the puncture $p(e)$ and Y has a cycle at the puncture $p(e')$. We then have a diagram

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(X) \arrow[r, "R_{p(e)}"] \arrow[d, "\Psi"'] & \mathcal{F}^{top}_{\infty}(S^1) \arrow[d, "\Psi"'] & \mathcal{F}^{top}_{\infty}(Y) \arrow[l, "R_{p(e')}"'] \arrow[d, "\Psi"'] \\
\mathcal{B}(G_i) \arrow[r, "R_e"] & \mathcal{B}(E) & \mathcal{B}(T) \arrow[l, "R_{e'}"']
\end{tikzcd} \tag{16}$$

where the horizontal arrows are restriction functors, and the vertical arrows are the equivalences given by the induction hypothesis. The fact that both squares commute is also part of the induction hypothesis. This equivalence of diagrams implies the equivalence of homotopy fiber products:

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(X \coprod_{S^1} Y) \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathcal{F}^{top}_{\infty}(Y) \arrow[d] \arrow[rr, bend left] & \mathcal{B}(G_i \coprod_E T) \arrow[r] \arrow[d] & \mathcal{B}(T) \arrow[d] \\
\mathcal{F}^{top}_{\infty}(X) \arrow[r] \arrow[rr, bend right] & \mathcal{F}^{top}_{\infty}(S^1) \arrow[rr, bend right] & \mathcal{B}(G_i) \arrow[r, "R_e"] & \mathcal{B}(E)
\end{tikzcd} \tag{17}$$

In the diagram above, the squares are homotopy fiber products, and the curved arrows are equivalences of categories. In particular, since $G_{i+1} \coprod_E T = G_i$, and $X \coprod_{S_1} Y$ is a skeleton for $\Sigma(G_{i+1})$, we have an equivalence

$$\Psi : \mathcal{F}^{top}_{\infty}(\Sigma(G_{i+1})) \to \mathcal{B}(G_{i+1}) \tag{18}$$

To complete the proof of the induction step, we must also consider the restriction functors to the punctures of $\Sigma(G_{i+1})$. On the $\mathcal{B}$-side, the infinite edges G_{i+1} correspond to infinite edges of G_i and T, minus the edges e and e' that we glue along. For each infinite edge e'' of G_{i+1}, we have a restriction functor $R_{e''} : \mathcal{B}(G_{i+1}) \to \mathcal{B}(E)$. This functor factors through either $\mathcal{B}(G_i)$ or $\mathcal{B}(T)$, according to whether e'' comes from G_i or T. On the $\mathcal{F}$-side, we have a corresponding restriction functor $R_{p(e'')} : \mathcal{F}^{top}_{\infty}(X \coprod_{S^1} Y) \to \mathcal{F}^{top}_{\infty}(S^1)$. Strictly speaking, the definition of this functor requires choosing a skeleton for $\Sigma(G_{i+1})$ that has a cycle at the puncture $p(e'')$, and $X \coprod_{S^1} Y$ may not have this property (and furthermore it is impossible for it to have this property with respect to every puncture simultaneously). The solution is Lemma 7.7, which says that we can modify either X or Y only in order to achieve that $X \coprod_{S^1} Y$ also has a cycle at $p(e'')$. Since this modification can be implemented on $X \coprod_{S^1} Y$ simultaneously, we find that the restriction to $p(e'')$ factors through the the closed restriction to either X or Y. If the puncture $p(e'')$ comes from $\Sigma(G_i)$, there is therefore a commutative diagram of closed restriction functors

$$\begin{tikzcd}
\mathcal{F}^{top}_{\infty}(X \coprod_{S^1} Y) \arrow[r, "R"] \arrow[dr, "R_{p(e'')}"'] & \mathcal{F}^{top}_{\infty}(X) \arrow[d, "R_{p(e'')}"] \\
& \mathcal{F}^{top}_{\infty}(S^1)
\end{tikzcd} \tag{19}$$

In the case that $p(e'')$ comes from T, the same diagram holds with Y in place of X in the upper-right node. Comparing the two sides, we have a diagram

$$\begin{tikzcd}
\mathcal{F}^{top}_\infty(X \coprod_{S^1} Y) \arrow[r, "R"] \arrow[dr, "R_{p(e'')}"'] \arrow[rr, bend left, "\Psi"] & \mathcal{F}^{top}_\infty(X) \arrow[d, "R_{p(e'')}"] \arrow[rr, bend left, "\Psi"] & \mathcal{B}(G_i \coprod_E T) \arrow[r] \arrow[dr] & \mathcal{B}(G_i) \arrow[d] \\
& \mathcal{F}^{top}_\infty(S^1) \arrow[rr, bend left, "\Psi"] & & \mathcal{B}(E)
\end{tikzcd} \tag{20}$$

In this diagram, the curved Ψ arrows (which are equivalences) commute form commutative squares with the horizontal and vertical arrows, and therefore they also form a commutative square with the diagonal arrows. This establishes the desired compatibility between restriction functors to infinite edges of G_{i+1} with restrictions to punctures of $\Sigma(G_{i+1})$.

Case of two edges: Let $e_1 \in G_i$ and $e'_1 \in T$ be on pair of edges being glued, and let $e_2 \in G_2$ and $e'_2 \in T$ be the other pair. Choose a skeleton X for $\Sigma(G_i)$ that has disjoint cycles at the punctures $p(e_1)$ and $p(e_2)$, and choose a skeleton Y for T that has disjoint cycles at the punctures $p(e'_1)$ and $p(e'_2)$ (this Y is necessarily a dumbbell graph). The argument proceeds as before, but we glue X to Y along $S^1 \coprod S^1$, and G_i to T along $E \coprod E$. Thus we have a diagram

$$\begin{tikzcd}
\mathcal{F}^{top}_\infty(X \coprod_{S^1 \coprod S^1} Y) \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathcal{F}^{top}_\infty(Y) \arrow[d] \arrow[rr, bend left] & \mathcal{B}(G_i \coprod_{E \coprod E} T) \arrow[r] \arrow[d] & \mathcal{B}(T) \arrow[d] \\
\mathcal{F}^{top}_\infty(X) \arrow[r] \arrow[rr, bend right] & \mathcal{F}^{top}_\infty(S^1 \coprod S^1) \arrow[rr, bend right] & \mathcal{B}(G_i) \arrow[r, "R_e"] & \mathcal{B}(E \coprod E)
\end{tikzcd} \tag{21}$$

where the two squares are homotopy fiber products and the curved arrows are equivalences.

It remains to analyze the restriction functors. If e'' is an infinite edge of $G_{i+1} = G_i \coprod_{E \coprod E} T$, then the restriction to e'' factors through restriction either to G_i or T. Similarly, we claim that the restriction from $\mathcal{F}^{top}_\infty(X \coprod_{S^1 \coprod S^1} Y)$ factors through restriction either to X or Y. The only issue here is that we may not be able to choose a skeleton that has disjoint cycles at three punctures simultaneously. This occurs when we consider the third puncture of $\Sigma(T)$, since Y is a dumbbell graph, or if X has only three punctures. On the other hand, the modification we need to do in order to produce a puncture at $p(e'')$ can be localized in a neighborhood of either X or Y inside $X \coprod_{S^1 \coprod S^1} Y$. Since an open restriction followed by a closed restriction is a closed restriction (see Proposition 5.14), it suffices to understand the closed restriction functor from a neighborhood of X or Y to the puncture. After restricting to a small enough neigborhood of X or Y, the closed restriction to X or Y consists then of merely removing some noncompact edges of the skeleton, and it makes no difference whether we do this before or after modifying the skeleton. Thus the restriction the puncture $p(e'')$ factors through restriction first to X or Y. The rest of the argument is the same as in the previous case. □

We are now ready to prove our main theorem. We use the notations of Section 3.1.1. Let $(X_{\mathcal{T}}, W_{\mathcal{T}})$ be a toric Calabi-Yau LG model, and let $\Sigma_{\mathcal{T}}$ be the mirror curve.

Theorem 8.3. *There is an equivalence of categories*

$$MF(X_{\mathcal{T}}, W_{\mathcal{T}}) \simeq Fuk^{top}(\Sigma_{\mathcal{T}}).$$

Proof. Let $G_{\mathcal{T}}$ be the tropical curve dual to the triangulation $\mathcal{T}$. Recall that $Fuk^{top}_{\infty}(\Sigma_{\mathcal{T}})$ denotes the Ind completion of $Fuk^{top}(\Sigma_{\mathcal{T}})$. By Theorem 3.9 and Theorem 8.2 there are equivalences

$$MF^{\infty}(X_{\mathcal{T}}, W_{\mathcal{T}}) \simeq \mathcal{B}(G_{\mathcal{T}}) \simeq Fuk^{top}_{\infty}(\Sigma_{\mathcal{T}}).$$

They restrict to an equivalence between the categories of compact objects inside $MF^{\infty}(X, W)$ and $Fuk^{\infty}_{top}(\Sigma_{\mathcal{T}})$

$$MF(X_{\mathcal{T}}, W_{\mathcal{T}}) \simeq Fuk^{top}(\Sigma_{\mathcal{T}}),$$

and this concludes the proof. ☐

Remark 8.4. Let $\Sigma_{\mathcal{T}}$ be an unramified cyclic cover of a punctured surface and let $Fuk^{wr}(\Sigma_{\mathcal{T}})$ be the wrapped Fukaya category. By [AAEKO] there is an equivalence

$$Fuk^{wr}(\Sigma_{\mathcal{T}}) \simeq MF(X_{\mathcal{T}}, f_{\mathcal{T}}).$$

Together with Theorem 8.3, this yields an equivalence

$$Fuk^{wr}(\Sigma_{\mathcal{T}}) \simeq Fuk^{top}(\Sigma_{\mathcal{T}}).$$

This establishes Kontsevich's claim [K], according to which the topological Fukaya category is equivalent the wrapped Fukaya category, for a large class of punctured Riemann surfaces. In her thesis Lee [Le] extends the results of [AAEKO] to all genera. This, combined with Theorem 8.3, gives a complete proof of Kontsevich's claim for punctured surfaces.

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