

Floer Cohomology in the Mirror of the Projective Plane and a Binodal Cubic Curve

by

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Abstract

We construct a family of Lagrangian submanifolds in the Landau–Ginzburg mirror to the projective plane equipped with a binodal cubic curve as anticanonical divisor. These objects correspond under mirror symmetry to the powers of the twisting sheaf $\mathcal{O}(1)$, and hence their Floer cohomology groups form an algebra isomorphic to the homogeneous coordinate ring. An interesting feature is the presence of a singular torus fibration on the mirror, of which the Lagrangians are sections. This gives rise to a distinguished basis of the Floer cohomology and the homogeneous coordinate ring parameterized by fractional integral points in the singular affine structure on the base of the torus fibration. The algebra structure on the Floer cohomology is computed using the symplectic techniques of Lefschetz fibrations and the TQFT counting sections of such fibrations. We also show that our results agree with the tropical analog proposed by Abouzaid–Gross–Siebert. Extensions to a restricted class of singular affine manifolds and to mirrors of the complements of components of the anticanonical divisor are discussed.

Thesis Supervisor: Denis Auroux
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Chapter 1

Introduction

Mirror symmetry is the name given to the phenomenon of deep, non-trivial, and sometimes even spectacular equivalences between the geometries of certain pairs of spaces. Such a pair (X, X^\vee) is called a mirror pair, and we say that X^\vee is the mirror to X and vice-versa. A byword for mirror symmetry is the equivalence, discovered by Candelas–de la Ossa–Green–Parkes [8] and proven mathematically by Givental [11] and Lian–Liu–Yau [26], between the Gromov–Witten theory of the quintic threefold $V_5 \subset \mathbb{P}^4$ and the theory of period integrals on a family of Calabi–Yau threefolds known as “mirror quintics.” Since this discovery, the study of mirror symmetry has expanded in many directions, both in physics and mathematics, allowing for generalization of the class of spaces considered, providing new algebraic ideas for how the equivalence ought to be conceptualized, and giving geometric insight into how a given space determines its mirror partner. In this introduction we provide some orientation and context that we hope will enable the reader to situate our work within this constellation of ideas.

1.1 Manifolds with effective anticanonical divisor and their mirrors

Due to their importance for supersymmetric string theory, the class of spaces originally considered in mirror symmetry were Calabi–Yau manifolds, the n -dimensional Kähler manifolds X for which the canonical bundle Ω_X^n is trivial. Generally speaking, the mirror to a compact Calabi–Yau manifold X is another compact Calabi–Yau manifold X^\vee of the same dimension. As a mathematical phenomenon, however, mirror symmetry has also been considered for other classes of manifolds. These include manifolds of general type (Ω_X^n ample), for which a proposal has recently been made by Kapustin–Katzarkov–Orlov–Yotov [20], and manifolds with an effective anticanonical divisor, which have a better developed theory and will concern us presently. In both of these latter cases the mirror is not a manifold of the same class.

Let X be an n -dimensional Kähler manifold with an effective anticanonical divisor. Let us actually choose a meromorphic $(n, 0)$ -form Ω that has only poles, and let the anticanonical divisor D be the polar locus of Ω . We regard D as part of the data, and write (X, D) for the pair. According to Hori–Vafa [18] and Givental, the mirror to (X, D) is a *Landau–Ginzburg model* (X^\vee, W) , consisting of a Kähler manifold X^\vee , together with a holomorphic function $W : X^\vee \rightarrow \mathbb{C}$, called the *superpotential*.

A large class of examples was derived by Hori–Vafa [18, §5.3] based on physical considerations. Let X be an n -dimensional toric Fano manifold, and let D be the complement of the open torus orbit (so that D is actually an ample divisor). Choose a polarization $\mathcal{O}_X(1)$ with corresponding moment polytope P , a lattice polytope in \mathbb{R}^n . For each facet F of P , let $\nu(F)$ to be the primitive integer inward-pointing normal vector, and let $\alpha(F)$ be such that $\langle \nu(F), x \rangle + \alpha(F) = 0$ is the equation for the hyperplane containing F . Then mirror Landau–Ginzburg model is given by

$$X^\vee = (\mathbb{C}^*)^n, \quad W = \sum_{F \text{ facet}} e^{-2\pi\alpha(F)} z^{\nu(F)}, \quad (1.1)$$

where $z^{\nu(F)}$ is a monomial in multi-index notation. In the case where X is toric but

not necessarily Fano, a similar formula for the mirror superpotential is expected to hold, which differs by the addition of “higher order” terms [10, Theorems 4.5, 4.6].

The Hori–Vafa formula contains the case of the projective plane \mathbb{CP}^2 with the toric boundary as anticanonical divisor. If x, y, z denote homogeneous coordinates, then D_{toric} can be taken to be $\{xyz = 0\}$, the union of the coordinate lines. We then have

$$X_{\text{toric}}^{\vee} = (\mathbb{C}^*)^2, \quad W_{\text{toric}} = z_1 + z_2 + \frac{e^{-\Lambda}}{z_1 z_2} \quad (1.2)$$

where Λ is a parameter that measures the cohomology class of the Kähler form ω on \mathbb{CP}^2 .

The example with which we are primarily concerned in this paper is also \mathbb{CP}^2 , but with respect to a different, nontoric boundary divisor. Consider the meromorphic $(n, 0)$ -form $\Omega = dx \wedge dz / (xz - 1)$, whose polar locus is the binodal cubic curve $D = \{xyz - y^3 = 0\}$. Thus $D = L \cup C$ is the union of a conic $C = \{xz - y^2 = 0\}$ and a line $\{y = 0\}$. The construction of the mirror to this pair (\mathbb{CP}^2, D) is due to Auroux [5], and we have

$$X^{\vee} = \{(u, v) \in \mathbb{C}^2 \mid uv \neq 1\}, \quad W = u + \frac{e^{-\Lambda} v^2}{uv - 1} \quad (1.3)$$

One justification for the claim that (1.1)–(1.3) are appropriate mirrors is found in the Strominger–Yau–Zaslow proposal, which expresses mirror symmetry geometrically in terms of dual torus fibrations. In the case of (1.3), this is actually how the construction proceeds.

1.2 Torus fibrations

An important insight into the geometric nature of mirror symmetry is the proposal by Strominger–Yau–Zaslow (SYZ) [35] to view two mirror manifolds X and X^{\vee} as dual special Lagrangian torus fibrations over the same base B . This relationship is called T-duality.

For our purposes, a Lagrangian submanifold L of a Kähler manifold X with mero-

morphic $(n, 0)$ -form Ω is called *special* of phase ϕ if $\text{Im}(e^{-i\phi}\Omega)|_L = 0$. Obviously this only makes sense in the complement of the polar locus D . The infinitesimal deformations of a special Lagrangian submanifold are given by $H^1(L; \mathbb{R})$, and are unobstructed [28]. If $L \cong T^n$ is a torus, $H^1(L; \mathbb{R})$ is an n -dimensional space, and in good cases the special Lagrangian deformations of L are all disjoint, and form the fibers of a fibration $\pi : X \setminus D \rightarrow B$, where B is the global parameter space for the deformations of L .

Assuming this, define the *complexified moduli space* of deformations of L to be the space \mathcal{M}_L consisting of pairs (L_b, \mathcal{E}_b) , where $L_b = \pi^{-1}(b)$ is a special Lagrangian deformation of L , and \mathcal{E}_b is a $U(1)$ local system on L_b . There is an obvious projection $\pi^\vee : \mathcal{M}_L \rightarrow B$ given by forgetting the local system. The fiber $(\pi^\vee)^{-1}(b)$ is the space of $U(1)$ local systems on the given torus L_b , which is precisely the dual torus L_b^\vee . In this sense, the fibrations π and π^\vee are dual torus fibrations, and the SYZ proposal can be taken to mean that the mirror X^\vee is precisely this complexified moduli space: $X^\vee = \mathcal{M}_L$. The picture is completed by showing that \mathcal{M}_L naturally admits a complex structure J^\vee , a Kähler form ω^\vee , and a holomorphic $(n, 0)$ -form Ω^\vee . One finds that Ω^\vee is constructed from ω , while ω^\vee is constructed from Ω , thus expressing the interchange of symplectic and complex structures between the two sides of the mirror pair. For details we refer the reader to [16],[5, §2].

However, this picture of mirror symmetry cannot be correct as stated, as it quickly hits upon a major stumbling block: the presence of singular fibers in the original fibration $\pi : X \setminus D \rightarrow B$. These singularities make it impossible to obtain the mirror manifold by a fiberwise dualization, and generate “quantum corrections” that complicate the T-duality prescription. Attempts to overcome this difficulty led to the remarkable work of Kontsevich and Soibelman [23, 24], and found a culmination in the work of Gross and Siebert [14, 15, 13] that implements the SYZ program in an algebro-geometric context. It is also this difficulty which motivates us to consider the case of $\mathbb{C}\mathbb{P}^2$ relative to a binodal cubic curve, where the simplest type of singularity arises.

In the case of X with effective anticanonical divisor D , we can see these corrections

in action if we include the superpotential W into the SYZ picture. As W is to be a function on X^\vee , which is naively \mathcal{M}_L , W assigns a complex number to each pair (L_b, \mathcal{E}_b) . This number is a count of holomorphic disks with boundary on L_b , of Maslov index 2, weighted by symplectic area and the holonomy of \mathcal{E}_b :

$$W(L_b, \mathcal{E}_b) = \sum_{\beta \in \pi_2(X, L_b), \mu(\beta)=2} n_\beta(L_b) \exp(-\int_\beta \omega) \text{hol}(\mathcal{E}_b, \partial\beta) \quad (1.4)$$

where $n_\beta(L_b)$ is the count of holomorphic disks in the class β passing through a general point of L_b .

In the toric case, $X \setminus D \cong (\mathbb{C}^*)^n$, and we the special Lagrangian torus fibration is simply the map $\text{Log} : X \setminus D \rightarrow \mathbb{R}^n$, $\text{Log}(z_1, \dots, z_n) = (\log |z_1|, \dots, \log |z_n|)$. This fibration has no singularities, and the above prescriptions work as stated. In the toric Fano case, we recover the Hori–Vafa superpotential (1.1).

However, in the case of $\mathbb{C}\mathbb{P}^2$ with the non-toric divisor D , the torus fibration one singular fiber, which is a pinched torus. The above prescription breaks down: one finds that the superpotential defined by (1.4) is not a continuous function on \mathcal{M}_L . This leads one to redefine X^\vee by breaking it into pieces and regluing so as to make W continuous. This is how Auroux [5] derives the mirror (1.3). We find that X^\vee also admits a special Lagrangian torus fibration with one singular fiber.

1.3 Affine manifolds

Moving back to the general SYZ picture, it is possible to distill the structure of a special Lagrangian torus fibration $\pi : X \rightarrow B$ into a structure on the base B : the structure of an *affine manifold*. This is a manifold with a distinguished collection of affine coordinate charts, such that the transition maps between affine coordinate charts lie in the group of affine transformations of Euclidean space: $\text{Aff}(\mathbb{R}^n) = \text{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n$. In fact, the base B inherits two affine structures, one from the symplectic form ω , and one from the holomorphic $(n, 0)$ -form Ω . The former is called the symplectic affine structure, and the latter is called the complex affine structure, since Ω determines

the complex structure (the vector fields X such that $\iota_X \Omega = 0$ are precisely those of type $(0, 1)$ with respect to the complex structure).

Let us recall briefly how the local affine coordinates are defined. For the symplectic affine structure, we choose a collection of loops $\gamma_1, \dots, \gamma_n$ that form a basis of $H_1(L_b; \mathbb{Z})$. Let $X \in T_b B$ be a tangent vector to the base, and take \tilde{X} be any vector field along L_b which lifts it. Then

$$\alpha_i(X) = \int_0^{2\pi} \omega_{\gamma_i(t)}(\dot{\gamma}_i(t), \tilde{X}(\gamma_i(t))) dt \quad (1.5)$$

defines a 1-form on B : since L_b is Lagrangian, the integrand is independent of the lift \tilde{X} , and α_i only depends on the class of γ_i in homology. In fact, the collection $(\alpha_i)_{i=1}^n$ forms a basis of $T_b^* B$, and there is a coordinate system $(y_i)_{i=1}^n$ such that $dy_i = \alpha_i$; these are the affine coordinates. This definition actually shows us that there is a canonical isomorphism $T_b^* B \cong H_1(L_b; \mathbb{R})$. This isomorphism induces an integral structure on $T_b^* B$: $(T_b^* B)_{\mathbb{Z}} \cong H_1(L_b; \mathbb{Z})$, which is preserved by all transition functions between coordinate charts. Thus, when an affine manifold arises as the base of a torus fibration in this way, the structural group is reduced to $\text{Aff}_{\mathbb{Z}}(\mathbb{R}^n) = \text{GL}(n, \mathbb{Z}) \rtimes \mathbb{R}^n$, the group of affine linear transformations with integral linear part.

The complex affine structure follows exactly the same pattern, only that we take $\Gamma_1, \dots, \Gamma_n$ to be $(n-1)$ -cycles forming a basis of $H_{n-1}(L_b; \mathbb{Z})$, and in place of ω we use $\text{Im}(e^{-i\phi}\Omega)$. Now we have an isomorphism $T_b^* B \cong H_{n-1}(L_b; \mathbb{R})$, or equivalently $T_b B \cong H_1(L_b; \mathbb{R})$, which induces the integral structure.

It is clear that these constructions of affine coordinates only work in the part of the fibration where there are no singular fibers. When singular fibers are present in the torus fibration, we simply regard the affine structure as being undefined at the singular fibers and call the resulting structure on the base a *singular affine manifold*.

In this paper, we are mainly interested in those affine manifolds that satisfy a stronger integrality condition, which requires the translational part of each transition function to be integral as well. We use the term *integral affine manifold* to denote an affine manifold whose structural group has been reduced to $\text{Aff}(\mathbb{Z}^n) = \text{GL}(n, \mathbb{Z}) \rtimes \mathbb{Z}^n$.

Such affine manifolds are “defined over \mathbb{Z} ,” and have an intrinsically defined lattice of integral points.

A natural class of subsets of an affine manifold B are the tropical subvarieties. These are certain piecewise linear complexes contained in B , which in some way correspond to holomorphic or Lagrangian submanifolds of the total space of the torus fibration. Tropical geometry has played a role in much work on mirror symmetry, particularly in the program of Gross and Siebert, and closer to this paper, in Abouzaid’s work on mirror symmetry for toric varieties [1, 2]. See [19] for a general introduction to tropical geometry. Though most of the methods in this paper are explicitly symplectic, tropical geometry does appear in two places, in Chapter 2, where we compute the tropicalization of the fiber of the superpotential as a motivation for our symplectic constructions, as well as in Chapter 5, where a class of tropical curves corresponding to holomorphic polygons is considered.

1.4 Homological mirror symmetry

Another major aspect of mirror symmetry that informs this paper is the homological mirror symmetry (HMS) conjecture of Kontsevich [21]. This holds that mirror symmetry can be interpreted as an equivalence of categories associated to the complex or algebraic geometry of X , and the symplectic geometry of X^\vee , and vice-versa. The categories which are appropriate depend somewhat on the situation, so let us focus on the case of a manifold X with anticanonical divisor D , and its mirror Landau–Ginzburg model (X^\vee, W) .

Associated to (X, D) , we take the derived category of coherent sheaves $D^b(\text{Coh } X)$, which is a standard object of algebraic geometry.

For (X^\vee, W) , we associate a Fukaya-type A_∞ -category $\mathcal{F}(X^\vee, W)$ whose objects are certain Lagrangian submanifolds of X^\vee , morphism spaces are generated by intersection points, and the A_∞ product structures are defined by counting pseudoholomorphic polygons with boundary on a collection of Lagrangian submanifolds. Our main reference for Floer cohomology and Fukaya categories is the book of Seidel [34].

The superpotential W enters the definition of $\mathcal{F}(X^\vee, W)$ by restricting the class of objects to what are termed *admissible Lagrangian submanifolds*. Originally, these were defined by Kontsevich [22] and Hori–Iqbal–Vafa [17] to be those Lagrangian submanifolds L , not necessarily compact, which outside of a compact subset are invariant with respect to the gradient flow of $\operatorname{Re}(W)$. An alternative formulation, due to Abouzaid [1, 2], trades the non-compact end for a boundary on a fiber $\{W = c\}$ of W , together with the condition that, near the boundary, the L maps by W to a curve in \mathbb{C} . A further reformulation, which is more directly related to the SYZ picture, replaces the fiber $\{W = c\}$ with the union of hypersurfaces $\bigcup_\beta \{z_\beta = c\}$, where z_β is the term in the superpotential (1.4) corresponding to the class $\beta \in \pi_2(X, \pi^{-1}(b))$, and admissibility means that near $\{z_\beta = c\}$, L maps by z_β to a curve in \mathbb{C} .

With these definitions, homological mirror symmetry amounts to an equivalence of categories $D^\pi \mathcal{F}(X^\vee, W) \rightarrow D^b(\operatorname{Coh} X)$, where D^π denotes the split-closed derived category of the A_∞ -category. This piece of mirror symmetry has been addressed many times [7, 6, 30, 1, 2, 9], including results for the projective plane and its toric mirror.

However, in this paper, we emphasize less the equivalences of categories themselves, and focus more on geometric structures which arise from a combination of the HMS equivalence with the SYZ picture. When dual torus fibrations are present on the manifolds in a mirror pair, one expects the correspondence between coherent sheaves and Lagrangian submanifolds to be expressible in terms of a Fourier–Mukai transform with respect to the torus fibration [25]. In particular, Lagrangian submanifolds $L \subset X^\vee$ that are sections of the torus fibration correspond to line bundles on X , and the Lagrangians we consider in this paper are of this type.

1.5 Distinguished bases

The homological formulation of mirror symmetry, particularly in conjunction with the SYZ proposal, gives rise to the expectation that, at least in favorable situations, the spaces of sections of coherent sheaves on X can be equipped with canonical bases.

To be more precise, suppose that $F : \mathcal{F}(X^\vee) \rightarrow D^b(X)$ is a functor implementing the HMS equivalence. Let $L_1, L_2 \in \text{Ob}(\mathcal{F}(X^\vee))$ be two objects of the Fukaya category supported by transversely intersecting Lagrangian submanifolds. Then

$$HF(L_1, L_2) \cong \text{RHom}(F(L_1), F(L_2)). \quad (1.6)$$

Suppose furthermore that the differential on the Floer cochain complex $CF(L_1, L_2)$ vanishes, so that $HF(L_1, L_2) \cong CF(L_1, L_2)$. As $CF(L_1, L_2)$ is defined to have a basis in bijection with the set of intersection points $L_1 \cap L_2$, one obtains a basis of $\text{RHom}(F(L_1), F(L_2))$ parameterized by the same set via the above isomorphisms. If \mathcal{F} is some sheaf of interest, and by convenient choice of L_1 and L_2 we can ensure $F(L_1) \cong \mathcal{O}_X$ and $F(L_2) \cong \mathcal{F}$, then we will obtain a basis for $H^i(X, \mathcal{F})$.

When E and E^\vee are mirror dual elliptic curves, this phenomenon is illustrated vividly by the work of Polishchuk–Zaslow [29]. Writing E^\vee as an S^1 fibration over S^1 , and taking two minimally intersecting sections L_1 and L_2 of this S^1 fibration, one obtains line bundles $F(L_1)$ and $F(L_2)$ on E . Supposing the line bundle $\mathcal{L} = F(L_2) \otimes F(L_1)^\vee$ to have positive degree, the basis of intersection points $L_1 \cap L_2$ corresponds to a basis of $\Gamma(E, \mathcal{L})$ consisting of theta functions.

Another illustration is the case of toric varieties and their mirror Landau-Ginzburg models (1.1), as worked out by Abouzaid [1, 2]. In this case, Abouzaid constructs a family of Lagrangian submanifolds $L(d)$ mirror to the powers of the ample line bundle $\mathcal{O}_X(d)$. These Lagrangian submanifolds are topologically discs with boundary on a level set of the superpotential, $W^{-1}(c)$ for some c . For $d > 0$, the Floer complex $CF^*(L(0), L(d))$ is concentrated in degree zero. Hence

$$CF^0(L(0), L(d)) = HF^0(L(0), L(d)) = H^0(X, \mathcal{O}_X(d)). \quad (1.7)$$

The basis of intersection points $L(0) \cap L(d)$ corresponds to the basis of characters of the algebraic torus $T = (\mathbb{C}^*)^n$ which appear in the T -module $H^0(X, \mathcal{O}_X(d))$.

In order to take a unified view on these examples, it is useful to interpret them in terms of integral affine or tropical geometry (as explicitly described in Abouzaid’s

work), and the intimately related Strominger–Yau–Zaslow perspective on mirror symmetry. The case of elliptic curves is easiest to understand, as both E and E^\vee may quite readily be written as special Lagrangian torus fibrations (in this dimension the fiber is an S^1) over the same base B , which in this case is a circle. The base has an integral affine structure as \mathbb{R}/\mathbb{Z} . The Lagrangians $L(d)$ are sections of this torus fibration, and their intersection points project precisely to the fractional integral points of the base B .

$$L(0) \cap L(d) \leftrightarrow B \left(\frac{1}{d}\mathbb{Z} \right) := \frac{1}{d}\text{-integral points of } B \quad (1.8)$$

The notation $B((1/d)\mathbb{Z})$ is in analogy with the functor-of-points notation.

The same formula (1.8) is valid in the case of toric varieties, where the base B is the moment polytope P of the toric variety X . Abouzaid interprets P as a subset of the base of the torus fibration on $X^\vee = (\mathbb{C}^*)^n$ (the fibration given by the Clifford tori), which moreover appears as a chamber bounded by a tropical variety corresponding to a level set $W^{-1}(c)$ of the superpotential.

An expert reading this will remark on an interesting feature of these constructions, which is that two affine structures seem to be in play at the same time.

On a Fano toric manifold, the symplectic affine structure on the base of the torus fibration is isomorphic to the interior of the moment polytope, while the complex affine structure is isomorphic to \mathbb{R}^n . In our case, the symplectic affine structure on the pair $(\mathbb{C}\mathbb{P}^2, D)$ is isomorphic to a bounded region B in \mathbb{R}^2 with a singular affine structure, while the complex affine structure is isomorphic to \mathbb{R}^2 equipped with a singular affine structure. Under mirror symmetry, the adjectives “symplectic” and “complex” are exchanged, so that the symplectic affine structure on X^\vee has infinite extent, while the complex affine structure is bounded. The integral points that parameterize our basis are integral for the complex affine structure on X^\vee , even though they come from (in our view) the symplectic geometry of this space, as intersection points of Lagrangian submanifolds.

Ongoing work of Gross–Hacking–Keel [12] seeks to extend these constructions to

other manifolds, such as K3 surfaces, using a purely algebraic and tropical framework. In this paper we are concerned with extensions to cases that are tractable from the point of view of symplectic geometry, although the tropical analog of our results is described in Chapter 5

1.6 Outline

In Chapter 2, we construct a tropicalization of the fiber of the superpotential $W^{-1}(c)$ over a large positive real value, with respect to a torus fibration with a single focus-focus singularity on the mirror of \mathbb{CP}^2 relative to the binodal cubic D . This gives a tropical curve in the base of our torus fibration. It bounds a compact region B in the base, which agrees with the symplectic affine base of the torus fibration on $\mathbb{CP}^2 \setminus D$. The purpose of this Chapter is to motivate the use of the singular affine manifold B as the basis for our main constructions.

In Chapter 3, we describe the main construction of the paper, which is a collection of Lagrangian submanifolds $\{L(d)\}_{d \in \mathbb{Z}}$ that is mirror to the collection $\mathcal{O}(d)$. The first step is to consider a family of symplectic forms on the space $X(B)$, which is a torus fibration over B , such that $X(B)$ forms a Lefschetz fibration over an annulus, and such that the boundary conditions for the Lagrangian submanifolds form flat sections of the Lefschetz fibration. The Lagrangian submanifolds $L(d)$ fiber over paths in the base of this Lefschetz fibration, and are defined by symplectic parallel transport of an appropriate Lagrangian in the fiber along this path. The construction also makes evident the correspondence between intersection points of the Lagrangian and fractional integral points of the base.

Chapter 4 forms the technical heart of the paper, where the computation of the product on the Floer cohomologies $HF^*(L(d_1), L(d_2))$ is accomplished using a degeneration argument. Here we reap the benefit of having constructed our Lagrangians carefully, as we are able to interpret the Floer products as counts of pseudo-holomorphic sections of the Lefschetz fibration. We use the TQFT counting pseudo-holomorphic sections of Lefschetz fibrations developed by Seidel to break the count

into simpler pieces, each of which can be computed rather explicitly using geometric techniques.

In Chapter 5, we consider the tropical analogue of the holomorphic triangles considered in Chapter 4. The definition of these curves comes from a recent proposal of Abouzaid, Gross and Siebert for a tropical Fukaya category associated to a singular affine manifold. Since we do not say anything about degenerating holomorphic polygons to tropical ones, we merely verify the equivalence by matching bases and computing on both sides.

The techniques developed in Chapters 3 and 4 actually apply to a larger but rather restricted class of 2-dimensional singular affine manifolds, where the main restriction is that all singularities have *parallel monodromy-invariant directions*. The generalization to these types of manifolds is discussed in Chapter 6.

In Chapter 7 we discuss an extension in another direction, which is to mirrors to complements of components of the anticanonical divisor, where the mirror theory involves wrapped Floer cohomology.

Chapter 2

The fiber of W and its tropicalization

2.1 Torus fibrations on $\mathbb{CP}^2 \setminus D$ and its mirror

Let $D = \{xyz - y^3 = 0\} \subset \mathbb{CP}^2$ be a binodal cubic curve. Both $\mathbb{CP}^2 \setminus D$ and its mirror $X^\vee = \{(u, v) \in \mathbb{C}^2 \mid uv \neq 1\}$ admit special Lagrangian torus fibrations. In fact, these spaces are diffeomorphic, each being \mathbb{C}^2 minus a conic. The torus fibrations are essentially the same on both sides, but we are interested in the *symplectic* affine structure associated to the fibration on $\mathbb{CP}^2 \setminus D$ and the *complex* affine structure associated to X^\vee .

The construction is taken from [5, §5]. Writing $D = \{xyz - y^3 = 0\}$, we see that $\mathbb{CP}^2 \setminus D$ is an affine algebraic variety with coordinates x and z , where $xz \neq 1$. Hence we can define a map $f : \mathbb{CP}^2 \setminus D \rightarrow \mathbb{C}^*$ by $f(x, z) = xz - 1$. This map is a Lefschetz fibration with critical point $(0, 0)$ and critical value -1 . The fibers are affine conics, and the map is invariant under the S^1 action $e^{i\theta}(x, z) = (e^{i\theta}x, e^{-i\theta}z)$ that rotates the fibers. Each fiber contains a distinguished S^1 -orbit, namely the vanishing cycle $\{|x| = |z|\}$. We can parameterize the other S^1 -orbits by the function $\delta(x, z)$ which denotes the signed symplectic area between the vanishing cycle and the orbit through (x, z) . The function δ is a moment map for the S^1 -action. Symplectic parallel transport in every direction preserves the circle at level $\delta = \lambda$, and so by

choosing any loop $\gamma \subset \mathbb{C}^*$, and $\lambda \in (-\Lambda, \Lambda)$ (where $\Lambda = \int_{\mathbb{CP}^1} \omega$ is the area of a line), we obtain a Lagrangian torus $T_{\gamma, \lambda} \subset \mathbb{CP}^2 \setminus D$. If we let $T_{R, \lambda}$ denote the torus at level λ over the circle of radius R centered at the origin in \mathbb{C}^* , we find that $T_{R, \lambda}$ is special Lagrangian with respect to the form $\Omega = dx \wedge dz / (xz - 1)$.

The torus fibration on X^\vee is essentially the same, except that the coordinates (x, z) are changed to (u, v) . For the rest of the paper, we denote by $w = uv - 1$ the quantity to which we project in order to obtain the Lagrangian tori $T_{R, \lambda}$ (and later the Lagrangian sections $L(d)$) as fibering over paths. For the time being, and in order to enable the explicit computations in section 2.3, we will equip X^\vee with the standard symplectic form in the (u, v) -coordinates, so the quantity $\delta(u, v)$ is the standard moment map $|u|^2 - |v|^2$. In summary, for X^\vee , we have $T_{R, \lambda} = \{(u, v) \mid |w| = |uv - 1| = R, |u|^2 - |v|^2 = \lambda\}$.

Each torus fibration has a unique singular fiber: $T_{1,0}$ which is a pinched torus.

Figure 2-1 shows several fibers of the Lefschetz fibration, with a Lagrangian torus that maps to a circle in the base. The two marked points in the base represent a Lefschetz critical value (filled-in circle), and puncture (open circle).

Before proceeding to study the fiber of W with respect to the torus fibration on X^\vee , we describe the symplectic affine structure on the base B of the torus fibration on $\mathbb{CP}^2 \setminus D$.

Proposition 2.1.1. *The affine structure on B has one singularity, around which the monodromy is a simple shear. B also has two natural boundaries, corresponding to when the torus degenerates onto the conic or the line, which form straight lines in the affine structure.*

Proof. This proposition can be extracted from the analysis in [5, §5.2]. The symplectic affine coordinates are the symplectic areas of disks in \mathbb{CP}^2 with boundary on $T_{R, \lambda}$. Let H denote the class of a line. The cases $R > 1$ and $R < 1$ are distinguished.

On the $R > 1$ side, we take $\beta_1, \beta_2 \in H_2(\mathbb{CP}^2, T_{R, \lambda})$ to be the classes of two sections over the disk bounded by the circle of radius R in the base, where β_1 intersects the

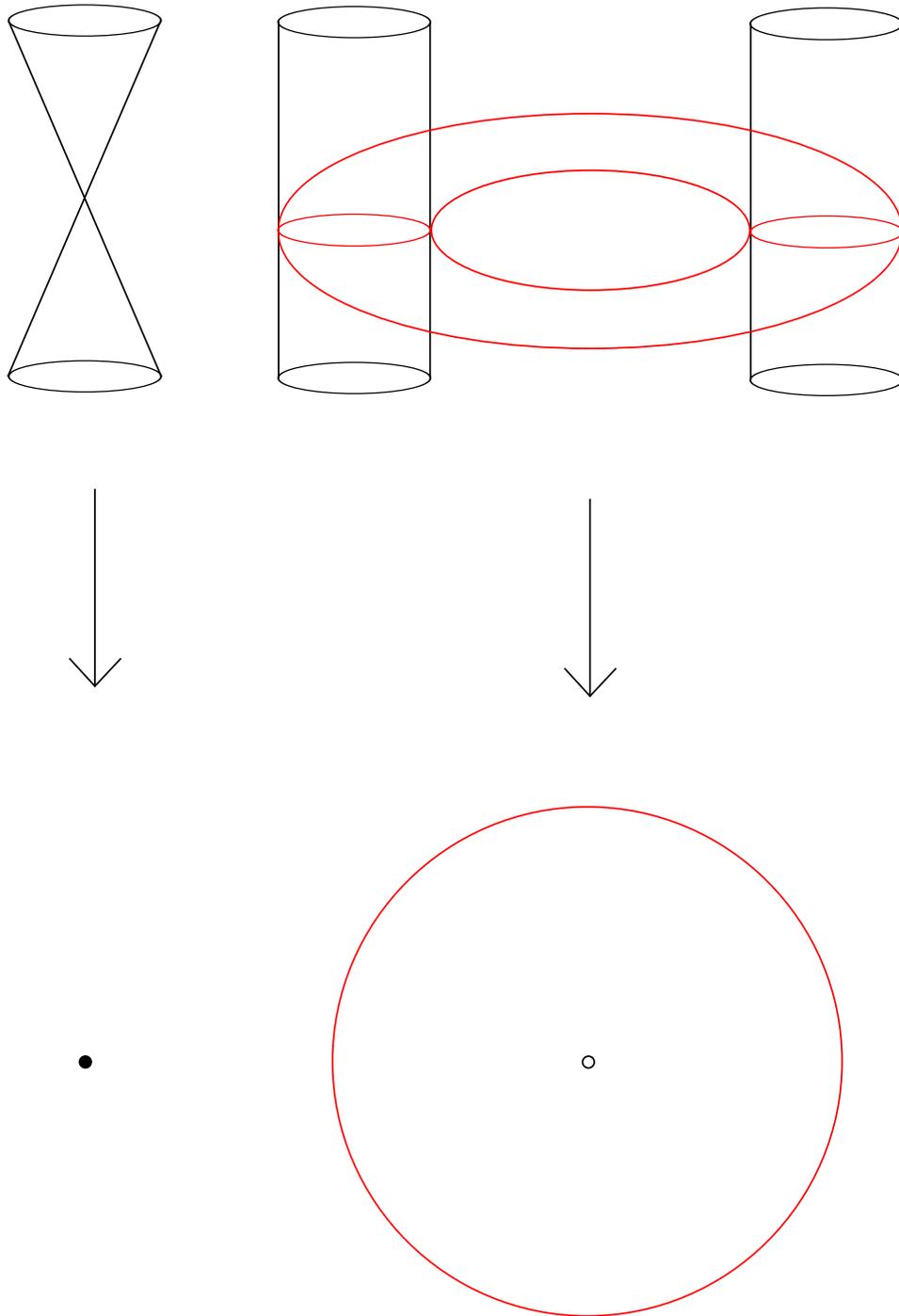


Figure 2-1: The Lefschetz fibration with a torus that maps to a circle.

z -axis and β_2 the x -axis. Then the torus fiber collapses onto line $\{y = 0\}$ when

$$\langle [\omega], H - \beta_1 - \beta_2 \rangle = 0. \quad (2.1)$$

On the $R < 1$ side, we take $\alpha, \beta \in H_2(\mathbb{CP}^2, T_{R,\lambda})$, where β is now the unique class of sections over the disk bounded by the circle of radius R , and α is the class of a disk connecting an S^1 -orbit to the vanishing cycle within the conic fiber and capping off with the thimble. The torus fiber collapses onto the conic $\{xz - y^2 = 0\}$ when

$$\langle [\omega], \beta \rangle = 0. \quad (2.2)$$

The two sides $R > 1$ and $R < 1$ are glued together along the wall at $R = 1$, but the gluing is different for $\lambda > 0$ than for $\lambda < 0$, leading to the monodromy. Let us take $\eta = \langle [\omega], \alpha \rangle$ and $\xi = \langle [\omega], \beta \rangle$ as affine coordinates in the $R < 1$ region. We continue these across the $\lambda > 0$ part of the wall using correspondence between homology classes:

$$\begin{aligned} \alpha &\leftrightarrow \beta_1 - \beta_2 \\ \beta &\leftrightarrow \beta_2 \\ H - 2\beta - \alpha &\leftrightarrow H - \beta_1 - \beta_2 \end{aligned} \quad (2.3)$$

Thus, in the $\lambda > 0$ part of the base, the conic appears as $\xi = 0$, while the line appears as

$$0 = \langle [\omega], H - 2\beta - \alpha \rangle = \Lambda - 2\xi - \eta \quad (2.4)$$

which is a line of slope of $-1/2$ with respect to the coordinates (η, ξ)

In the $\lambda < 0$ part of the base, we instead use

$$\begin{aligned} \alpha &\leftrightarrow \beta_1 - \beta_2 \\ \beta &\leftrightarrow \beta_1 \\ H - 2\beta + \alpha &\leftrightarrow H - \beta_1 - \beta_2 \end{aligned} \quad (2.5)$$

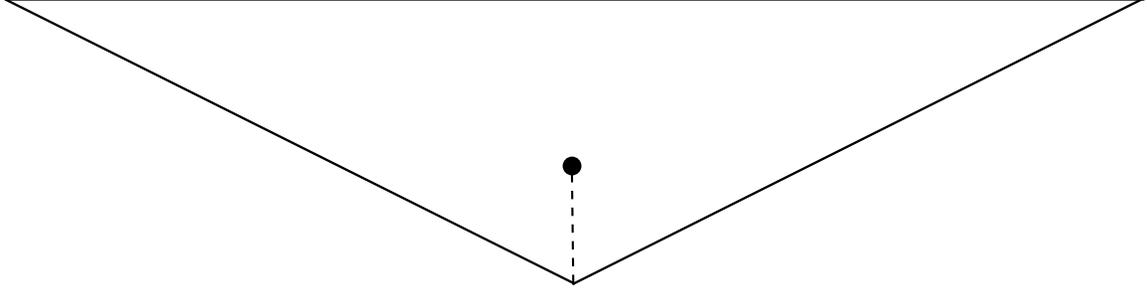


Figure 2-2: The affine manifold B .

Hence in this region the conic appears as $\xi = 0$ again, while the line appears as

$$0 = \langle [\omega], H - 2\beta + \alpha \rangle = \Lambda - 2\xi + \eta \quad (2.6)$$

which is a line of slope $1/2$ with respect to the coordinates (η, ξ) .

The discrepancy between the two gluings represents the monodromy. As we pass from $\{R > 1, \lambda > 0\} \rightarrow \{R < 1, \lambda > 0\} \rightarrow \{R < 1, \lambda < 0\} \rightarrow \{R > 1, \lambda < 0\} \rightarrow \{R > 1, \lambda > 0\}$, the coordinates (η, ξ) under go the transformation $(\eta, \xi) \rightarrow (\eta, \xi - \eta)$, which is indeed a simple shear. \square

The goal of the rest of this section is to find the same affine manifold B (that comes from symplectic structure of $\mathbb{C}\mathbb{P}^2 \setminus D$) embedded in the complex geometry of the Landau–Ginzburg model. We find that B is a subset of the base of the torus fibration on X^\vee , equipped with the *complex* affine structure, which is bounded by a particular tropical curve, the tropicalization of the fiber of W .

Figure 2-2 shows the affine manifold B . The marked point is a singularity of the affine structure, and the dotted line is a branch cut in the affine coordinates. Going around the singularity counterclockwise, the monodromy of the tangent bundle is given by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

2.2 The topology of the map W

A direct computation shows that the superpotential W given by (1.3) has three critical points

$$\text{Crit}(W) = \{(v = e^{\Lambda/3} e^{2\pi i(n/3)}, w = 1) \mid n = 0, 1, 2\}, \quad (2.7)$$

and corresponding critical values

$$\text{Critv}(W) = \{3e^{-\Lambda/3} e^{-2\pi i(n/3)} \mid n = 0, 1, 2\}. \quad (2.8)$$

As expected, $\text{Critv}(W)$ is the set of eigenvalues of quantum multiplication by $c_1(T\mathbb{C}\mathbb{P}^2)$ in $QH^*(\mathbb{C}\mathbb{P}^2)$, that is, multiplication by $3h$ in the ring $\mathbb{C}[h]/\langle h^3 = e^{-\Lambda} \rangle$.

Proposition 2.2.1. *Any regular fiber $W^{-1}(c) \subset X^\vee$ is a twice-punctured elliptic curve.*

Proof. In the (u, v) coordinates, $W^{-1}(c)$ is defined by the equation

$$u + \frac{e^{-\Lambda} v^2}{uv - 1} = c, \quad (2.9)$$

$$u(uv - 1) + e^{-\Lambda} v^2 = c(uv - 1). \quad (2.10)$$

This is an affine cubic plane curve, and it is disjoint from the affine conic $V(uv - 1)$. Here $V(\dots)$ denotes the vanishing locus. It is smooth as long as c is a regular value.

The projective closure of $W^{-1}(c)$ in (u, v) coordinates is given by the homogeneous equation (with ξ as the third coordinate)

$$u(uv - \xi^2) + e^{-\Lambda} v^2 \xi = c\xi(uv - \xi^2). \quad (2.11)$$

This is a projective cubic plane curve, hence elliptic, and it intersects the line at infinity $\{\xi = 0\}$ when $u^2 v = 0$. So it is tangent to the line at infinity at $(u : v : \xi) = (0 : 1 : 0)$ and intersects it transversely at $(u : v : \xi) = (1 : 0 : 0)$. Hence the affine curve is the projective curve minus these two points.

□

Remark 1. The function W above is to be compared to the “standard” superpotential for $\mathbb{C}\mathbb{P}^2$, namely,

$$W = x + y + \frac{e^{-\Lambda}}{xy} \tag{2.12}$$

corresponding to the choice of the toric boundary divisor, a union of three lines, as anticanonical divisor. This W has the same critical values, and its regular fibers are all thrice-punctured elliptic curves. Hence smoothing the anticanonical divisor to the union of a conic and a line corresponds to compactifying one of the punctures of $W^{-1}(c)$. This claim can be interpreted in terms of T-duality.

2.3 Tropicalization in a singular affine structure

Now we will describe a method for constructing what we consider to be a tropicalization of the fiber of the superpotential.

In the conventional picture of tropicalization, one considers a family of subvarieties of an algebraic torus $V_t \subset (\mathbb{C}^*)^n$. The map $\text{Log} : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n$ given by $\text{Log}(z_1, \dots, z_n) = (\log |z_1|, \dots, \log |z_n|)$ projects these varieties to their amoebas $\text{Log}(V_t)$, and the rescaled limit of these amoebas is the tropicalization of the family V_t . The tropicalization is also given as the non-archimedean amoeba of the defining equation of V_t , as shown in various contexts by various people (Kapranov, Rullgård, Speyer-Sturmfels).

We take the view that this map $\text{Log} : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n$ is the projection map of a special Lagrangian fibration. Its fibers are the tori defined by fixing the modulus of each complex coordinate. These tori are Lagrangian with respect to the standard symplectic form, and they are special with respect to the holomorphic volume form

$$\Omega_{\text{toric}} = \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}, \tag{2.13}$$

which has logarithmic poles along the coordinate hyperplanes in \mathbb{C}^n .

In the case at hand, we have a pencil of curves $W^{-1}(c)$ in $X^\vee \cong \mathbb{C}^2 \setminus V(uv - 1)$. The total space X^\vee must now play the role that $(\mathbb{C}^*)^2 = \mathbb{C}^2 \setminus V(uv)$ plays in ordinary tropical geometry. The holomorphic volume form is

$$\Omega = \frac{du \wedge dv}{uv - 1} = \frac{du \wedge dv}{v}. \quad (2.14)$$

Differentiating the defining equation $uv = 1 + w$ and substituting gives the other formulas

$$\Omega = \frac{du}{u} \wedge \frac{dw}{w}, \quad \text{when } u \neq 0, \quad (2.15)$$

$$\Omega = -\frac{dv}{v} \wedge \frac{dw}{w}, \quad \text{when } v \neq 0. \quad (2.16)$$

The special Lagrangian fibration on X^\vee to consider is constructed in [5]. The fibers are the tori

$$T_{R,\lambda} = \{(u, v) \in X^\vee \mid |uv - 1| = R, |u|^2 - |v|^2 = \lambda\}, \quad (R, \lambda) \in (0, \infty) \times (-\infty, \infty), \quad (2.17)$$

and the fiber $T_{1,0}$ is a pinched torus. Thus (R, λ) are coordinates on the base of this fibration. But they are not affine coordinates, which must be computed from the flux of the holomorphic volume form. Due to the simple algebraic form of this fibration, it is possible to find an integral representation of the (complex) affine coordinates explicitly.

Proposition 2.3.1. *In the subset of the base where $R < 1$, a set of affine coordinates is*

$$\begin{aligned} \eta &= \log |w| = \log R \\ \xi &= \frac{1}{2\pi} \int_{T_{R,\lambda} \cap \{u \in \mathbb{R}_+\}} \log |u| d \arg(w) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \log \left(\frac{\lambda + \sqrt{\lambda^2 + 4 \cdot |1 + Re^{i\theta}|^2}}{2} \right) d\theta \end{aligned} \quad (2.18)$$

Another set is

$$\begin{aligned}
\eta &= \log |w| = \log R \\
\psi &= \frac{1}{2\pi} \int_{T_{R,\lambda} \cap \{v \in \mathbb{R}_+\}} \log |v| d \arg(w) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \log \left(\frac{-\lambda + \sqrt{\lambda^2 + 4 \cdot |1 + R e^{i\theta}|^2}}{2} \right) d\theta
\end{aligned} \tag{2.19}$$

These coordinates satisfy

$$\xi + \psi = 0. \tag{2.20}$$

Proof. The general procedure for computing affine coordinates from the flux of the holomorphic volume form is as follows: we choose, over a local chart on the base, a collection of $(2n-1)$ -manifolds $\{\Gamma_i\}_{i=1}^n$ in the total space X such that the torus fibers T_b intersect each Γ_i in an $(n-1)$ -cycle, and such that these $(n-1)$ -cycles $T_b \cap \Gamma_i$ form a basis of $H_{n-1}(T_b; \mathbb{Z})$. The affine coordinates $(y_i)_{i=1}^n$ are defined up to constant shift by the property that

$$y_i(b') - y_i(b) = \frac{1}{2\pi} \int_{\Gamma_i \cap \pi^{-1}(\gamma)} \text{Im } \Omega \tag{2.21}$$

where γ is any path in the local chart on the base connecting b to b' . Because Ω is holomorphic, it is closed, and hence this integral does not depend on the choice of γ .

To get the coordinate system (2.18), we start with the submanifolds defined by

$$\Gamma_1 = \{w \in \mathbb{R}_+\}, \quad \Gamma_2 = \{u \in \mathbb{R}_+\} \tag{2.22}$$

The intersection $\Gamma_1 \cap T_{R,\lambda}$ is a loop on $T_{R,\lambda}$; the function $\arg(u)$ gives a coordinate on this loop (briefly, $w \in \mathbb{R}_+$ and $|w| = R$ determine uv , along with $|u|^2 - |v|^2 = \lambda$ this determines the $|u|$ and $|v|$; the only parameter left is $\arg(u)$ since $\arg(v) = -\arg(u)$), and we declare the loop to be oriented so that $-d \arg(u)$ restricts to a positive volume form on it (in the course of this computation we introduce several minus signs solely

for convenience later on). Using (2.15) we see

$$\mathrm{Im} \Omega = d \arg(u) \wedge d \log |w| + d \log |u| \wedge d \arg(w) \quad (2.23)$$

Using the fact that $\arg(w)$ is constant on Γ_1 , we see that for any path γ in the subset of the base where $R < 1$ connecting $b = (R, \lambda)$ to $b' = (R', \lambda')$, we have

$$\int_{\Gamma_1 \cap \pi^{-1}(\gamma)} \mathrm{Im} \Omega = \int_{\Gamma_1 \cap \pi^{-1}(\gamma)} d \arg(u) \wedge d \log |w|. \quad (2.24)$$

But $d \arg(u) \wedge d \log |w| = d(-\log |w| d \arg(u))$, so the integral above equals

$$\int_{\Gamma_1 \cap T_{b'}} -\log |w| d \arg(u) - \int_{\Gamma_1 \cap T_b} -\log |w| d \arg(u) = 2\pi(\log R' - \log R) \quad (2.25)$$

(the minus signs within the integrals are absorbed by the orientation convention for $\Gamma_1 \cap T_b$). Thus $\eta = \log R$ is the affine coordinate corresponding to Γ_1 .

The intersection $\Gamma_2 \cap T_{R,\lambda}$ is a loop on $T_{R,\lambda}$; together with the loop $\Gamma_1 \cap T_{R,\lambda}$ it gives a basis of $H_1(T_{R,\lambda}; \mathbb{Z})$. The function $\arg(w)$ gives a coordinate on this loop, and we orient the loop so that $d \arg(w)$ restricts to a positive volume form. Using (2.23), the fact that $\arg(u)$ is constant on Γ_2 , and the same reasoning as above, we see that

$$\int_{\Gamma_2 \cap \pi^{-1}(\gamma)} \mathrm{Im} \Omega = \int_{\Gamma_2 \cap T_{b'}} \log |u| d \arg(w) - \int_{\Gamma_2 \cap T_b} \log |u| d \arg(w). \quad (2.26)$$

Thus $\xi = \frac{1}{2\pi} \int_{\Gamma_2 \cap T_b} \log |u| d \arg(w)$ is the affine coordinate correspond to Γ_2 .

To arrive at the second formula for ξ , we must solve for $|u|$ in terms of R, λ , and $\theta = \arg(w)$. The equations $uv = 1 + Re^{i\theta}$ and $|u|^2 - |v|^2 = \lambda$ imply $|u|^4 - \lambda|u|^2 = |1 + Re^{i\theta}|^2$. Solving for $|u|^2$ by the quadratic formula and taking logarithms gives the result.

To get the coordinate system (2.19), we must consider now the subset $\Gamma'_2 = \{v \in \mathbb{R}_+\}$. This intersects each fiber in a loop along which $\arg(w)$ is once again a coordinate. Due to the minus sign in (2.16), we must orient the loop so that $-d \arg(w)$ is a positive volume form in order to get the formula we want. Otherwise, the derivation of ψ is

entirely analogous to the the derivation of ξ from Γ_2 .

There are two ways to prove (2.20). Either one adds the explicit integral representations of ξ and ψ , uses the law of logarithms, much cancellation, and the fact

$$\int_0^{2\pi} \log |1 + Re^{i\theta}| d\theta = 0, \text{ for } R < 1, \quad (2.27)$$

(an easy application of the Cauchy Integral Formula), or one uses the corresponding relation in the homology group $H_1(T_{R,\lambda}; \mathbb{Z})$ that

$$\Gamma_2 \cap T_{R,\lambda} + \Gamma'_2 \cap T_{R,\lambda} = 0 \quad (2.28)$$

where these loops are oriented as in the previous paragraphs, which shows that $\xi + \psi$ is constant, and one checks a particular value. \square

Proposition 2.3.2. *In the subset of the base where $R > 1$, the expressions (2.18) and (2.19) still define affine coordinate systems. However, now we have the relation*

$$\xi + \psi = \eta. \quad (2.29)$$

Hence the pair (ξ, ψ) also defines a coordinate system in the region $R > 1$.

Proof. The computation of the coordinates should go through verbatim in this case. As for (2.29), one could use the homological relationship between the loops, and this would give the equation up to an additive constant. Or one could simply take the sum of ξ and ψ , which reduces to

$$\frac{1}{2\pi} \int_0^{2\pi} \log |1 + Re^{i\theta}| d\theta = \log R, \text{ for } R > 1. \quad (2.30)$$

For this we use the identity

$$\log |1 + Re^{i\theta}| = \log |Re^{i\theta}(1 + R^{-1}e^{-i\theta})| = \log R + \log |1 + R^{-1}e^{-i\theta}|; \quad (2.31)$$

the integral of the second term vanishes by (2.27) since $R^{-1} < 1$.

□

Remark 2. We have seen that, in the (u, w) coordinates, the holomorphic volume form is standard. If the special Lagrangian fibration were also standard, the affine coordinates would be $(\log |u|, \log |w|)$. Proposition 2.3.1 shows that, while $\log |w|$ is still an affine coordinate (reflecting the fact that there is still an S^1 -symmetry), the other affine coordinate is the average value of $\log |u|$ along a loop in the fiber. Thus the coordinates η , ξ , and ψ correspond approximately to the log-norms of w , u , and v respectively. Furthermore, we see that when $|\lambda|$ is large, the approximations $\xi \approx \log |u|$ and $\psi \approx \log |v|$ become better.

Remark 3. Propositions 2.3.1 and 2.3.2 determine the monodromy around the singular point (at $\eta = \xi = \psi = 0$) of our affine base, and show that the affine structure is in fact integral.

We now describe the tropicalization process for the fiber of the superpotential. Consider the curve $W^{-1}(e^{\epsilon\Lambda})$:

$$W = u + \frac{e^{-\Lambda}v^2}{w} = e^{\epsilon\Lambda} \tag{2.32}$$

The tropicalization corresponds to the limit $\Lambda \rightarrow \infty$, or $t = e^{-\Lambda} \rightarrow 0$.

Now consider any of the coordinate systems (u, w) , (v, w) , or (u, v) , each of which is only valid in certain subset of X^\vee . Corresponding to each we have Log maps $(\log |u|, \log |w|)$, $(\log |v|, \log |w|)$, and $(\log |u|, \log |v|)$. We can therefore define an amoeba by $A_t(W^{-1}(e^{\epsilon\Lambda})) = \text{Log}(W^{-1}(e^{\epsilon\Lambda}))/\log t$. We can also take the tropical (nonarchimedean) amoeba of the curve 2.32 by substituting $t = e^{-\Lambda}$ and taking t as the generator of the valuation ideal, which gives us a graph in the base. As usual, the tropical amoeba is the Hausdorff limit of the amoebas $A_t(W^{-1}(e^{\epsilon\Lambda}))$, as $t = e^{-\Lambda} \rightarrow 0$.

Furthermore, as we take $t = e^{-\Lambda} \rightarrow 0$, the amoebas $A_t(W^{-1}(e^{\epsilon\Lambda}))$ move farther away from the singularity, where the approximations $\xi = \log |u|$ and $\psi = \log |v|$ hold with increasing accuracy. This means that at the level of tropical amoebas, we can actually identify the tropical coordinates ξ and $\log |u|$, ψ and $\log |v|$, in appropriate regions on the base of the torus fibration, while $\eta = \log |w|$ holds exactly everywhere.

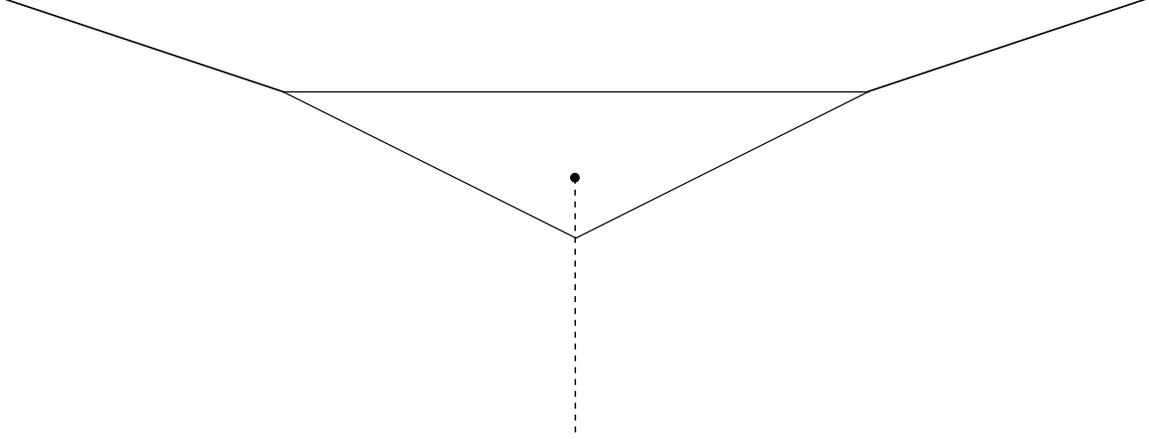


Figure 2-3: The tropical fiber of W .

By taking parts of each tropical amoeba where these identifications of tropical coordinates is valid, we find that the tropical amoebas computed in the three coordinate systems actually match up to give a single tropical curve, which we denote T_ϵ .

Proposition 2.3.3. *For $\epsilon > 0$, T_ϵ is a trivalent graph with two vertices, a cycle of two finite edges, and two infinite edges.*

For $(-1/3) < \epsilon < 0$, T_ϵ is a trivalent graph with three vertices, a cycle of three finite edges, two infinite edges and one edge connecting a vertex to the singular point of the affine structure.

Figure 2-3 shows the tropicalization of the fiber of W .

Remark 4. Note that in both cases of proposition 2.3.3, the topology of the tropical curve corresponds to that of a twice-punctured elliptic curve. The limit value $\epsilon = -1/3$ corresponds to the critical values of W .

Proposition 2.3.4. *For $\epsilon > 0$, the complement of T_ϵ has a bounded component that is an integral affine manifold with singularities that is isomorphic, after rescaling, to the base B of the special Lagrangian fibration on $\mathbb{C}\mathbb{P}^2 \setminus D$ with the affine structure coming from the symplectic form.*

Remark 5. This proposition is another case of the phenomenon, described in Abouzaid's

paper [1], that for toric varieties, the bounded chamber of the fiber of the superpotential is isomorphic to the moment polytope. In fact, this is part of the general SYZ picture in this context. In the general case of a manifold X with effective anticanonical divisor D , the boundary of the base of the torus fibration on $X \setminus D$ corresponds to a torus fiber collapsing onto D , a particular class of holomorphic disks having vanishing area, and the corresponding term of the superpotential having unit norm. On the other hand, the tropicalization of the fiber of the superpotential has some parts corresponding to one of the terms having unit norm, and it is expected that these bound a chamber which is isomorphic to the base of the original torus fibration.

Chapter 3

Symplectic constructions

Let B the affine manifold which is the bounded chamber of the tropicalization of the fiber of W . In this section we construct a symplectic structure on the manifold $X(B)$, which is a torus fibration over B , together with a Lefschetz fibration $w : X(B) \rightarrow X(I)$, where $X(I)$ is an annulus. Corresponding to the two sides of B , and hence to the two terms of $W = u + e^{-\Lambda}v^2/(uv - 1)$, we have horizontal boundary faces $\partial^h X(B)$, along each of which the symplectic connection defines a foliation. Choosing a leaf of the foliation on each face defines a boundary condition (corresponding to the fiber of W) for our Lagrangian submanifolds $\{L(d)\}_{d \in \mathbb{Z}}$, which are constructed as fibering over paths in the base of the Lefschetz fibration.

The motivation for these constructions is existence of the map $w = uv - 1 : X^\vee \rightarrow \mathbb{C}^*$, which is a Lefschetz fibration with general fiber an affine conic and a single critical value. The tori in the SYZ fibration considered in Chapter 2 fiber over loops in this projection, so it is natural to attempt to use it to understand as much of the geometry as possible. In particular it will allow us to apply the techniques of [34], [31], [30].

3.1 Monodromy associated to a Hessian metric

Let B be an affine manifold, which we will take to be a subset of \mathbb{R}^2 . Let η and ξ denote affine coordinates. Suppose that $\eta : B \rightarrow \mathbb{R}$ is a submersion over some interval $I \subset \mathbb{R}$, and that the fibers of this map are connected intervals. For our purposes, we

may consider the case where B is a quadrilateral, bounded on two opposite sides by line segments of constant η (the *vertical* boundary $\partial^v B$), and on the other two sides by line segments that are transverse to the projection to η (the *horizontal* boundary $\partial^h B$).

This setup is a tropical model of a Lefschetz fibration. We regard the affine manifolds B and I as the complex affine structures associated to torus fibrations on spaces $X(B)$ and $X(I)$. Clearly, $X(I)$ is an annulus, and $X(B)$ is a subset of a complex torus with coordinates w and z such that $\eta = \log |w|$ and $\xi = \log |z|$. The map $\eta : B \rightarrow I$ is a tropical model of the map $w : X(B) \rightarrow X(I)$.

In this situation, the most natural way to prescribe a Kähler structure on $X(B)$ is through a Hessian metric on the base B . This is a metric g such that locally $g = \text{Hess } K$ for some function $K : B \rightarrow \mathbb{R}$, where the Hessian is computed with respect to an affine coordinate system. If $\pi : X(B) \rightarrow B$ denotes the projection, then $\phi = K \circ \pi$ is a real potential on $X(B)$, and the positivity of $g = \text{Hess } K$ corresponds to the positivity of the real closed $(1, 1)$ -form $\omega = dd^c \phi$. Explicitly, if y_1, \dots, y_n are affine coordinates corresponding to complex coordinates z_1, \dots, z_n , then

$$g = \sum_{i,j=1}^n \frac{\partial^2 K}{\partial y_i \partial y_j} dy_i dy_j \tag{3.1}$$

$$\omega = dd^c \phi = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^n \frac{\partial^2 K}{\partial y_i \partial y_j} \frac{dz_i}{z_i} \wedge \frac{d\bar{z}_j}{\bar{z}_j} \tag{3.2}$$

This Kähler structure is invariant under the S^1 -action $e^{i\theta}(z, w) = (e^{i\theta}z, w)$ that rotates the fibers of the map $w : X(B) \rightarrow X(I)$.

Once we have a Hessian metric on B and a Kähler structure on $X(B)$, the fibration $w : X(B) \rightarrow X(I)$ has a symplectic connection. The base of the fibration is the annulus $X(I)$ so there is monodromy around loops there. The symplectic connection may be computed as follows: Let $X \in T_{(z,w)}X(B)$ denote a tangent vector. Let $Y \in \ker dw$ denote the general vertical vector. The relation defining the horizontal

distribution is $\omega(X, Y) = 0$, or,

$$\begin{aligned}
0 &= \left\{ K_{\eta\eta} \frac{dw}{w} \wedge \frac{d\bar{w}}{\bar{w}} + K_{\eta\xi} \left(\frac{dw}{w} \wedge \frac{d\bar{z}}{\bar{z}} + \frac{dz}{z} \wedge \frac{d\bar{w}}{\bar{w}} \right) + K_{\xi\xi} \frac{dz}{z} \wedge \frac{d\bar{z}}{\bar{z}} \right\} (X, Y) \\
&= K_{\eta\xi} \left(\frac{dw(X)}{w} \frac{d\bar{z}(Y)}{\bar{z}} - \frac{d\bar{w}(X)}{\bar{w}} \frac{dz(Y)}{z} \right) + K_{\xi\xi} \left(\frac{dz(X)}{z} \frac{d\bar{z}(Y)}{\bar{z}} - \frac{dz(Y)}{z} \frac{d\bar{z}(X)}{\bar{z}} \right) \\
&= \left(K_{\eta\xi} \frac{dw(X)}{w} + K_{\xi\xi} \frac{dz(X)}{z} \right) \frac{d\bar{z}(Y)}{\bar{z}} - \text{complex conjugate}
\end{aligned} \tag{3.3}$$

Since $dz(Y)$ can have any phase, this shows that the quantity in parentheses on the last line must vanish:

$$d \log z(X) = -\frac{K_{\eta\xi}}{K_{\xi\xi}} d \log w(X) \tag{3.4}$$

Tropically, this formula has the following interpretation: In the (η, ξ) coordinates, the vertical tangent space is spanned by the vector $(0, 1)$. The g -orthogonal to this space is spanned by the vector $(K_{\xi\xi}, -K_{\eta\xi})$, whose slope with respect to the affine coordinates is the factor $-K_{\eta\xi}/K_{\xi\xi}$ appearing in the formula for the connection.

Consider the parallel transport of the connection around the loop $|w| = R$, which is a generator of $\pi_1(X(I))$. This loop cannot be seen tropically. As w traverses the path $R \exp(it)$, the initial condition $(z, w) = (r \exp(i\theta), R)$ generates the solution $(r \exp(i\theta + (-K_{\eta\xi}/K_{\xi\xi})it), R \exp(it))$, where the expression $-K_{\eta\xi}/K_{\xi\xi}$ is constant along the solution curve. As a self-map of the fiber over $w = R$, this monodromy transformation maps circles of constant $|z|$ to themselves, but rotates each by the phase $2\pi(-K_{\eta\xi}/K_{\xi\xi})$.

We now consider the behavior of the symplectic connection near the horizontal boundary $\partial^h X(B)$. A natural assumption to make here is that $\partial^h B$ is g -orthogonal to the fibers of the map $\eta : B \rightarrow I$, and we assume this from now on. Let F be a component of $\partial^h B$. Since F is a straight line segment g -orthogonal to the fibers of η , the function $-K_{\eta\xi}/K_{\xi\xi}$ is constant on F and equal to its slope, which we denote $\sigma = \sigma_F$. We assume this slope to be rational. The part of $X(B)$ lying over F is defined by the condition $\log |z| = \sigma \log |w| + C$. Let $w = w_0 \exp(\rho(t) + i\phi(t))$ describe an arbitrary curve in the base annulus $X(I)$. If (z_0, w_0) is an initial point that lies

over F , then

$$z = z_0 \exp \{ \sigma(\rho(t) + i\phi(t)) \}, w = w_0 \exp(\rho(t) + i\phi(t)) \quad (3.5)$$

is a path in $X(B)$ that lies entirely over F , and which by virtue of this fact also solves the symplectic parallel transport equation. Thus the part of $\partial^h X(B)$ that lies over F is fibered by flat sections of the fibration, namely $(w/w_0)^\sigma = (z/z_0)$ where $\pi(z_0, w_0) \in F$. Take note that σ is merely rational, so these flat sections may actually be multisections.

Examples of the Hessian metrics with the above properties may be constructed by starting with the function

$$F(x, y) = x^2 + \frac{y^2}{x} \quad (3.6)$$

$$\text{Hess } F = \begin{pmatrix} 2 + 2\frac{y^2}{x^3} & -2\frac{y}{x^2} \\ -2\frac{y}{x^2} & 2\frac{1}{x} \end{pmatrix} \quad (3.7)$$

$$\frac{-F_{xy}}{F_{yy}} = \frac{y}{x} \quad (3.8)$$

Thus the families of lines $x = c$ and $y = \sigma x$ form an orthogonal net for Hess F . By taking x and y to be shifts of the affine coordinates η and ξ on B , we can obtain a Hessian metric on B such that the vertical boundary consists lines of the form $x = c$, while the horizontal boundary consists of lines of the form $y = \sigma x$.

3.2 Focus-focus singularities and Lefschetz singularities

Now we consider the case where the affine structure on B contains a focus-focus singularity, and the monodromy invariant direction of this singularity is parallel to the fibers of the map $\eta : B \rightarrow I$. The goal is to construct a symplectic manifold

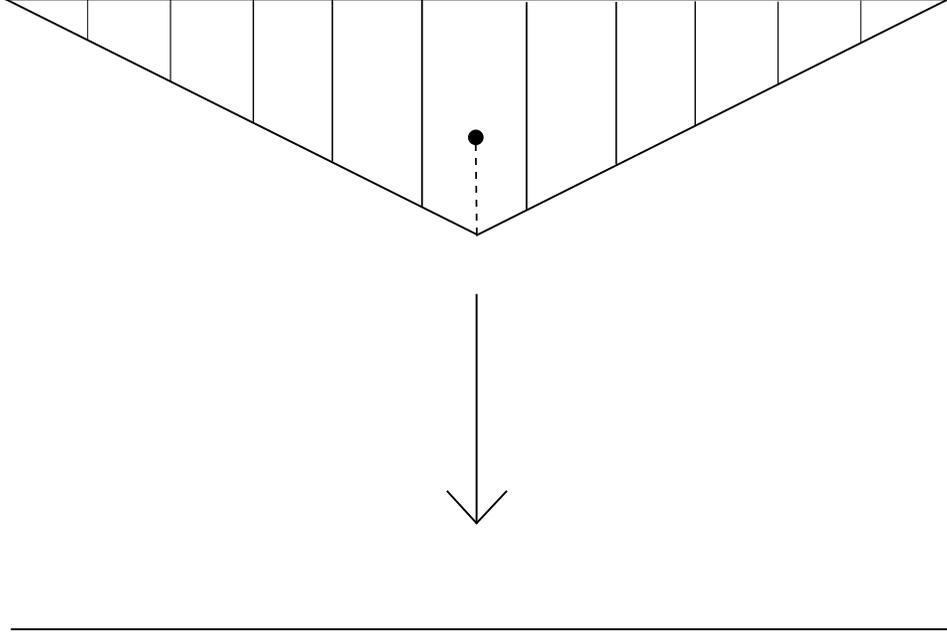


Figure 3-1: The fibration $B \rightarrow I$.

$X(B)$, along with a Lefschetz fibration $w : X(B) \rightarrow X(I)$. The critical point of the Lefschetz fibration occurs at the singular point of the focus-focus singularity, while on either side the singularity, the symplectic structure is of the form considered in section 3.1.

Figure 3-1 shows the projection $B \rightarrow I$, with the fibers drawn as vertical lines.

So let B be an affine manifold with a single focus-focus singularity, and $\eta : B \rightarrow I$ a globally defined affine coordinate. Suppose B has vertical boundary consisting of fibers of η , as before, and suppose that the horizontal boundary consists of line segments of rational slope. If we draw the singular affine structure with a branch cut, one side will appear straight while the other appears bent, though the bending is compensated by the monodromy of the focus-focus singularity.

Suppose for convenience that the singularity occurs at $\eta = 0$. Divide the base B into regions $B_{-\epsilon} = \eta^{-1}(-\infty, -\epsilon]$ and $B_{+\epsilon} = \eta^{-1}[\epsilon, \infty)$. On these affine manifolds we may take the Hessian metrics and associated Kähler forms considered in section 3.1. Hence we get a fibration with symplectic connection over the disjoint union of two annuli: $w : X(B_{-\epsilon} \amalg B_{+\epsilon}) \rightarrow X(I_{-\epsilon} \amalg I_{+\epsilon})$

First we observe that it is possible to connect the two sides by going “above” and “below” the focus-focus singularity. In other words, we consider two bands connecting $B_{-\epsilon}$ to $B_{+\epsilon}$ near the two horizontal boundary faces. Since the boundary faces are straight in the affine structure, we can extend the Hessian metric in such a way that the boundary faces are still orthogonal to the fibers of η , and so the portion of $X(B)$ lying over these faces is foliated by the symplectic connection.

Now we look at the fibration over the two annuli $X(I_{-\epsilon}) \amalg X(I_{+\epsilon}) \subset \mathbb{C}^*$. Choose a path connecting these two annuli, along the positive real axis, say. By identifying the fibers over the end points, the fibration extends over this path. By thickening the path up to a band and filling in the fibers over the band, we get a Lefschetz fibration over a surface which is topologically a pair of pants. If we also include the portions we filled in near the horizontal boundary, then we have a manifold with boundary, where one part of the boundary lies over the horizontal and vertical boundary of B , while the other is topologically an S^3 , which we fill in with a local model of a Lefschetz singularity. In order for this to make sense, we need the monodromy around the loop in the base being filled in to be a Dehn twist.

This can be seen by comparing the monodromy transformations around the loops in $X(I_{-\epsilon})$ and $X(I_{+\epsilon})$. Let z_- and z_+ denote complex coordinates on $X(B_{-\epsilon})$ and $X(B_{+\epsilon})$ corresponding to the “ ξ ” direction as in the previous section. These coordinates match up on one side of the singularity, but on the side where the branch cut has been placed they do not. Let F_1 and F_0 denote the top and bottom faces of $\partial^h B$ respectively, and suppose that F_0 is split into two parts F_{0+} and F_{0-} by the branch cut. Associated to each of these we have a slope σ_F .

Suppose we traverse a loop in $X(I_{-\epsilon})$ in the negative sense followed by a loop in $X(I_{+\epsilon})$ in the positive sense, connecting these paths through the band, and as we do this we measure the *difference* between the amounts of phase rotation in the z_- and z_+ coordinates along the top and bottom horizontal boundaries under parallel transport, encoding this as an overall twisting. As we transport around the negative loop in $X(I_{-\epsilon})$, the z_- coordinates on $\pi^{-1}(F_1)$ and $\pi^{-1}(F_{0-})$ twist relatively to each other by an amount $-(\sigma_{F_1} - \sigma_{F_{0-}})$, while on the other side the z_+ coordinates twist

by an amount $(\sigma_{F_1} - \sigma_{F_{0+}})$. Overall, we have a twisting of $\sigma_{F_{0-}} - \sigma_{F_{0+}}$: due to the form of the monodromy, this always equals -1 , and this is what we expect for the Dehn twist. The top and bottom boundaries are actually fixed under the monodromy transformation because the fibration is trivial there.

This allows us to fill in the fibration with a standard fibration with a single Lefschetz singularity whose vanishing cycle is the equatorial circle on the cylinder fiber. Since this local model is symmetric under the S^1 -action which rotates the fibers, choosing an S^1 -invariant gluing allows us to define a symplectic S^1 -action on $X(B)$ which rotates the fibers of $w : X(B) \rightarrow X(I)$.

Since the total space is S^1 -symmetric, we can construct the Lagrangian tori as in section 2.1, by taking circles of constant $|w|$ in the base and S^1 -orbits in the fiber. These actually coincide with the tori found in $X(B_{-\epsilon} \amalg B_{+\epsilon})$ as fibers of the projection to B , so this construction extends the torus fibrations on $X(B_{-\epsilon} \amalg B_{+\epsilon})$ to all of $X(B)$.

Since this construction is local on the base $X(I)$, the construction extends in an obvious way to the situation where several focus-focus singularities with parallel monodromy-invariant directions are present in B , and these monodromy invariant directions are vertical for the map $\eta : B \rightarrow I$. The result is again a fibration over an annulus $X(I)$, with a Lefschetz singularity for each focus-focus singularity.

If we restrict to the case where B is the manifold appearing in the mirror of (\mathbb{CP}^2, D) , The fibration $w : X(B) \rightarrow X(I)$ has the property that the horizontal boundary $\partial^h X(B)$ is the union of two faces $(\partial^h X(B))_1$ and $(\partial^h X(B))_0$ corresponding to F_1 and F_0 , the top and bottom faces of B . Each face is foliated by the symplectic connection:

- The leaves of the foliation on $(\partial^h X(B))_1$ are single-valued sections of the w -fibration, and in terms of the superpotential $W = u + e^{-\Lambda} v^2 / (uv - 1)$, they correspond to the curves defined by the first term: $u = \text{constant}$.
- The leaves of the foliation on $(\partial^h X(B))_0$ are two-valued sections of the w -fibration, and in terms of W they correspond to the curves defined by the

second term: $v^2/(uv - 1) = \text{constant}$.

Remark 6. The symplectic forms constructed in this section have many desirable properties, are convenient for computation, and apparently make mirror symmetry valid for the examples considered. However, a fuller understanding of the SYZ philosophy would most likely single out a smaller family of symplectic forms, though it is somewhat unclear what such forms should be (see the remark after Conjecture 3.10 in [5]). Since the affine structure coming from the complex structure of $\mathbb{CP}^2 \setminus D$ has infinite extent, one could ask for symplectic forms which become infinite as we approach the boundary of B . It seems reasonable that such forms can be constructed using the ideas presented here, but since we want to consider Lagrangian submanifolds with boundary conditions, and for technical convenience, it is easier to use symplectic forms that are finite at the horizontal boundaries of $X(B)$.

3.3 Lagrangians fibered over paths

Recall that the base of the Lefschetz fibration is the annulus $X(I) = \{R^{-1} \leq |w| \leq R\}$ with a critical value at $w = -1$. For visualizing the Lagrangians it is convenient to assume that the symplectic connection is flat throughout the annuli $X(I_{-\epsilon}) = \{R^{-1} \leq |w| \leq e^{-\epsilon}\}$, $X(I_{+\epsilon}) = \{e^\epsilon \leq |w| \leq R\}$, as well as through a band along the positive real axis joining these annuli.

3.3.1 The zero-section

The first step is to construct the Lagrangian submanifold $L(0) \subset X(B)$, which we will use as a zero-section and reference point through out the paper.

We take the path in the base $\ell(0) \subset X(I)$ which runs along the positive real axis. In a band around $\ell(0)$, the symplectic fibration is trivial, and we lift $\ell(0) \subset X(I)$ to $L(0) \subset X(B)$ by choosing a path in the fiber cylinder, and taking $L(0)$ to be the product. If we want to be specific, we could take the factor in the fiber to be the positive real locus of the coordinates z_- or z_+ .

Once $L(0)$ is chosen, it selects a leaf of each foliation on each boundary face, namely those leaves where its boundary lies. Call these leaves Σ_0 and Σ_1 (bottom and top respectively). Clearly we could have chosen these leaves first and then constructed $L(0)$ accordingly.

3.3.2 The degree d section

We can now use $L(0)$ as a reference to construct the other Lagrangians $L(d)$. Let $\ell(d)$ be a base path, with the same end points and midpoint as $\ell(0)$, and which winds d times (relative to $\ell(0)$) in $X(I_{-\epsilon})$ and also d times in $X(I_{+\epsilon})$. The winding of $\ell(d)$ is clockwise as we go from smaller to larger radius. As for the behavior in the fiber, we take $L(d)$ to coincide with $L(0)$ in the fibers over the common endpoints of $\ell(0)$ and $\ell(d)$. This then serves as the initial condition for parallel transport along $\ell(d)$, and we take $L(d)$ to be the manifold swept out by this parallel transport. Because the boundary curves Σ_0 and Σ_1 are parallel, $L(d)$ has boundary on these same curves everywhere.

The Lagrangian submanifold $L(d)$ is indeed a section of the torus fibration. If $T_{R,\lambda}$ is the torus over the circle $\{|w| = R\}$ at height λ , then since $\ell(d)$ intersects $\{|w| = R\}$ at one point, there is exactly one fiber of $w : X(B) \rightarrow X(I)$, where $L(d)$ and $T_{R,\lambda}$ intersect. Since $L(d)$ intersects each S^1 -orbit in that fiber once, we find that $L(d)$ and $T_{R,\lambda}$ indeed intersect once.

We now explain in what sense these Lagrangians are admissible. The relevant notion of admissibility is the one found in [5, §7.2], where admissibility with respect to a reducible hypersurface whose components correspond to the terms of the superpotential is discussed. In our case, we have two components Σ_0 and Σ_1 , and the admissibility condition is that, near Σ_i , the holomorphic function z_i such that $\Sigma_i = \{z_i = 1\}$ satisfies $z_i|L \in \mathbb{R}$. The Lagrangian $L(d)$ will have this property if the monodromy near Σ_1 is actually trivial, while the monodromy near Σ_0 is a rigid rotation by π . Otherwise, we can only say that the phase of z_i varies within a small range near Σ_i . Either way, we will ultimately end up perturbing the Lagrangians so that this weaker notion of admissibility holds.

The notion of admissibility is more important for understanding what happens over the endpoints of the base path $\ell(d)$. This point actually represents the corner of the affine base B , where the two boundary curves Σ_0 and Σ_1 intersect (that the symplectic form we chose was infinite at the corner explains why we don't see this intersection from the point of view of the fibration $w : X(B) \rightarrow X(I)$). This means that near the corner, the same part of the Lagrangian has to be admissible for both boundaries, and this forces the Lagrangian to coincide with $L(0)$ there.

Figure 3-2 depicts $L(0)$, $L(1)$, and $L(2)$. The lower portion of the figure shows the base: the straight line is $\ell(0)$, while the spirals are $\ell(1)$ and $\ell(2)$. The marked point is the Lefschetz critical value. The upper portion of the figure shows the five fibers where $\ell(0)$ and $\ell(2)$ intersect.

3.3.3 A perturbation of the construction

The Lagrangians $L(d)$ constructed above intersect each other on the boundary of $X(B)$, and in particular it is not clear whether such intersection points are supposed count toward the Floer cohomology. However, it is possible to perturb the construction in a conventional way so as to push all intersection points which should count toward Floer cohomology into the interior of $X(B)$.

The general convention is that we perturb the Lagrangians near the boundary so that the boundary intersection points have degree 2 for Floer cohomology, and then we forget the boundary intersection points.

Remark 7. If we do not care whether the Lagrangian actually has boundary on Σ_0 and Σ_1 , but is rather only near these boundaries, we can further use a small perturbation near the boundary to actually destroy the intersection points we wish to forget about. This is the point of view used in section 4.2.

The perturbation appropriate for computing $HF^*(L(0), L(d))$ with $d > 0$ is the following: we perturb the base path $\ell(d)$ near the end points by creating a new intersection point in the interior, in addition to the one on the boundary. We also perturb the part of $L(d)$ over the fiber at $w = R^{-1}$, which was the initial condition

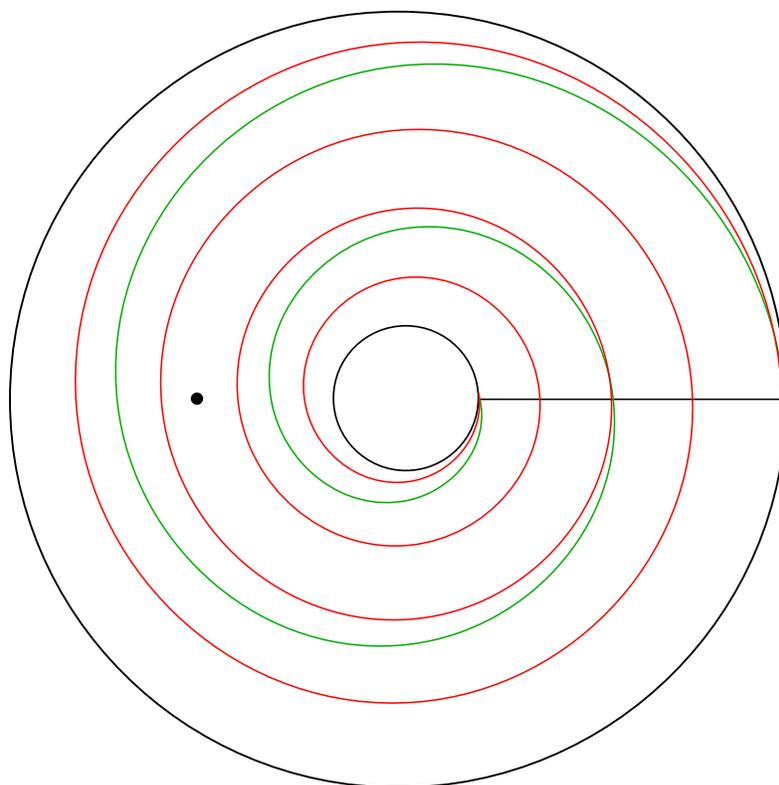
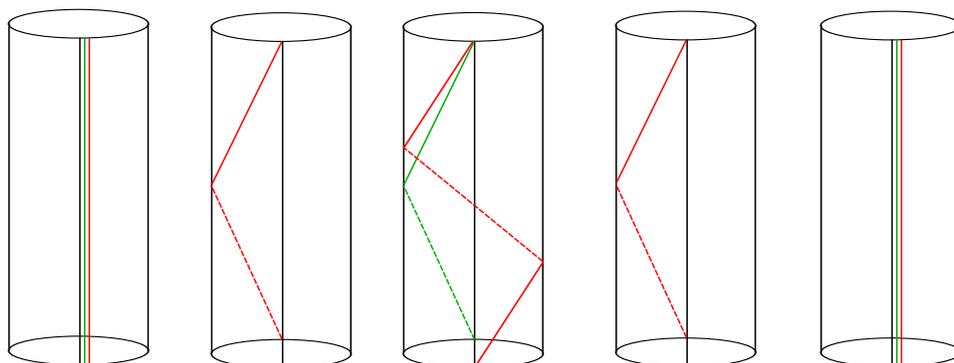


Figure 3-2: The Lagrangians $L(0)$, $L(1)$, and $L(2)$.

for the parallel transport construction of $L(d)$, so that rather than coinciding $L(d)$ intersects $L(0)$ once in the interior as well as on the boundary, and at this intersection point, the tangent space of $L(d)$ is a small clockwise rotation of the tangent space of $L(0)$. After parallel transport this will ensure the intersections of $L(0)$ and $L(d)$ over other points of $\ell(0) \cap \ell(d)$ are transverse as well. With an appropriate choice of complex volume form Ω for the purpose of defining gradings on Floer complexes, all of the interior intersection points will have degree 0 when regarded as morphisms going from $L(0)$ to $L(d)$, while the intersection points on the boundary have degree 2.

Hence in computing morphisms from $L(d)$ to $L(0)$ with $d > 0$, we perform the perturbation in the opposite direction. This does not create new intersections in the interior, and the boundary intersection points are forgotten, so there are actually fewer generators of $CF^*(L(d), L(0))$ than there are for $CF^*(L(0), L(d))$ when $d > 0$.

3.3.4 Intersection points and integral points

Using the perturbed Lagrangians $L(d)$, we are ready to work out the bijection between the intersection points of $L(0)$ and $L(d)$ with $d > 0$, regarded as morphisms from $L(0)$ to $L(d)$, and the $(1/d)$ -integral points of B .

We start at the intersection point of $\ell(0)$ and $\ell(d)$ at near the inner radius of the annulus $X(I)$. In the fiber over this point there is one intersection point. As we transport around the inner part of the annulus, we pick up half-twists in the fiber, which increases the number of intersection points by one after every two turns in the base. So for example after two turns, if we look in the fiber over the point where $\ell(0)$ and $\ell(d)$ intersect, there will be two intersections. This pattern continues until $\ell(d)$ reaches the middle radius and starts winding around the other side of the Lefschetz singularity, where the pattern reverses.

If we assign the rational numbers

$$[-1, 1] \cap (1/d)\mathbb{Z} = \{-1, -(d-1)/d, \dots, -1/d, 0, 1/d, \dots, 1\}$$

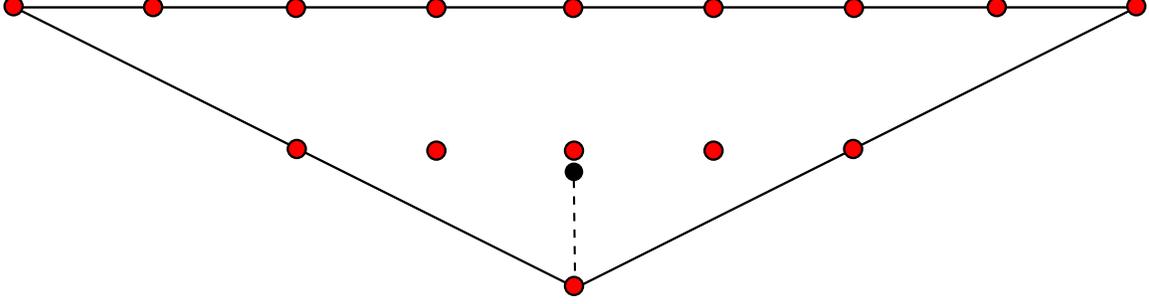


Figure 3-3: The $1/4$ -integral points of B .

to the intersection points of $\ell(0)$ and $\ell(d)$, then we see that over the point indexed by a/d the number of intersection points in the fiber is $1 + \left\lfloor \frac{d-|a|}{2} \right\rfloor$.

It is clear that if we scale B so that the top face has affine length 2, the $1/d$ integral points of B are organized by the projection $\eta : B \rightarrow I$ into columns indexed by $[-1, 1] \cap (1/d)\mathbb{Z}$, where the column over a/d has $1 + \left\lfloor \frac{d-|a|}{2} \right\rfloor$ of the $1/d$ -integral points.

A convenient way to index the intersection points in each column is by their distance from the top of the fiber. So in the column over a/d , we have intersections indexed by $i \in \{0, 1, \dots, \left\lfloor \frac{d-|a|}{2} \right\rfloor\}$ which lie at distances $i / \left(\frac{d-|a|}{2} \right)$ from the top of the fiber.

Definition 1. For $a \in \{-d, \dots, d\}$, and $i \in \{0, 1, \dots, \left\lfloor \frac{d-|a|}{2} \right\rfloor\}$, let $q_{a,i} \in L(n) \cap L(n+d)$ which lies in the column indexed by a/d , and which lies at a distance $i / \left(\frac{d-|a|}{2} \right)$ from the top of the fiber.

We can also observe at this point that

$$|L(0) \cap L(d)| = \left| B \left(\frac{1}{d}\mathbb{Z} \right) \right| = \frac{(d+2)(d+1)}{2} = \dim H^0(\mathbb{CP}^2, \mathcal{O}_{\mathbb{CP}^2}(d)) \quad (3.9)$$

thus verifying mirror symmetry at the level of the Hilbert polynomial.

Figure 3-3 shows the points of $B(\frac{1}{4}\mathbb{Z})$. representing the basis of morphisms $L(d) \rightarrow L(d+4)$.

3.3.5 Hamiltonian isotopies

There is an alternative way to express the relationship between $L(d)$ and $L(0)$, which is by a Hamiltonian isotopy. The simplest way to express this is to work in the base and the fiber separately. We start with $L(0)$, which is contained in a piece of the fibration which has a preferred trivialization. Hence we can apply the flow of a Hamiltonian function H_f on the fiber which generates the configuration of $1 + \lfloor d/2 \rfloor$ intersection points we need to have in the “central” fiber over $w = 1$. Then we apply the flow of a Hamiltonian function $H_b = f(|w|)$ which generates the desired twisting of the base paths while fixing the central fiber. Due to the monodromy of the fibration, this will unwind the Lagrangian in the fibers and give us a manifold isotopic to $L(d)$. Note that during the intermediate times of this isotopy the Lagrangian will not satisfy the boundary condition at Σ_0 and Σ_1 (in the first part of the isotopy), or it will not satisfy our condition at the endpoints of the base path corresponding to the corners of B . However, at the end of the isotopy these conditions are restored.

Chapter 4

A degeneration of holomorphic triangles

Since we have set up our Lagrangians as fibered over paths, a holomorphic triangle with boundary on the Lagrangians composed with the projection is a holomorphic triangle in the base, which is an annulus, with boundary along the corresponding paths. The triangles that are most interesting are those that pass over the critical value $w = -1$ (possibly several times). In general, such triangles are immersed in the annulus, and, after passing the the universal cover of the annulus, are embedded. Hence, we can regard such triangles as sections over a triangle in the base of a Lefschetz fibration having as base a strip with a \mathbb{Z} -family of critical values, and as fiber a cylinder. Once this is done, a TQFT for counting sections of Lefschetz fibrations set up by Seidel [34] can be brought to bear.

We consider the deformation of the Lefschetz fibration over the triangle where the critical values bubble out along one of the sides. At the end of this degeneration, we count sections of a trivial fibration over a $(k+3)$ -gon, along with sections of k identical fibrations, each having a disk with one critical value and one boundary marked point. Each of these fibrations is equipped with a Lagrangian boundary condition given by following the degeneration of the original Lagrangian submanifolds. The sections of the trivial fibration over the $(k+3)$ -gon can be reduced to counts in the fiber, while the counts of the k other parts are identical, and can be deduced most expediently

from the long exact sequence for Floer cohomology as it applies to the Lagrangian submanifolds of the fiber; in fact the count we are looking for is almost the same as one of the maps in this exact sequence.

4.1 Triangles as sections

Let $q_1 \in HF^0(L(0), L(n))$ and $q_2 \in HF^0(L(n), L(n+m))$ be two degree zero morphisms whose Floer product $\mu^2(q_2, q_1)$ we wish to compute. Suppose that $p \in HF^0(L(0), L(n+m))$ contributes to this product. Then there are holomorphic triangles in $X(B)$ connecting the points q_1, q_2, p , with boundary on $L(0), L(n), L(n+m)$.

Naturally, we consider the projection of such a triangle to the base by $w : X(B) \rightarrow X(I)$. This yields a 2-chain on the base with boundary on the corresponding base paths $\ell(0), \ell(n), \ell(n+m)$, and whose corners lie at the points $w(q_1), w(q_2), w(p)$ over which our original intersection points lie.

For the next step it is convenient to pass to the universal cover of the base. Let $\tilde{X}(I)$ denote infinite strip which is the universal cover of the annulus, and let $\tilde{w} : \tilde{X}(B) \rightarrow \tilde{X}(I)$ denote the induced fibration. When drawing pictures in the base $\tilde{X}(I)$, we can represent it as $[-1, 1] \times \mathbb{R}$, with the infinite direction drawn vertically. With this convention, the path $\ell(d)$ lifts to a \mathbb{Z} -family of paths which have slope $-d$.

Figure 4-1 shows the universal cover of $X(I)$, with the base paths for $L(0), L(1)$, and $L(2)$.

The choice of lift of q_1 determines a lift of $L(0)$ and $L(n)$, which then determines a lift of q_2 and of $L(n+m)$, which in turn determines where the lift of any p must lie. By looking at the slopes of the base paths $\ell(0), \ell(n), \ell(n+m)$ involved, we obtain the following proposition:

Proposition 4.1.1. *In the terminology of Definition 1, Suppose that $q_1 = q_{a,i}$ lies in the fiber indexed by a/n , and that $q_2 = q_{b,j}$ lies in the fiber indexed by b/m . Then any p contributing to the product is $q_{a+b,h}$ for some h , that is, it lies in the fiber indexed by $(a+b)/(n+m)$*

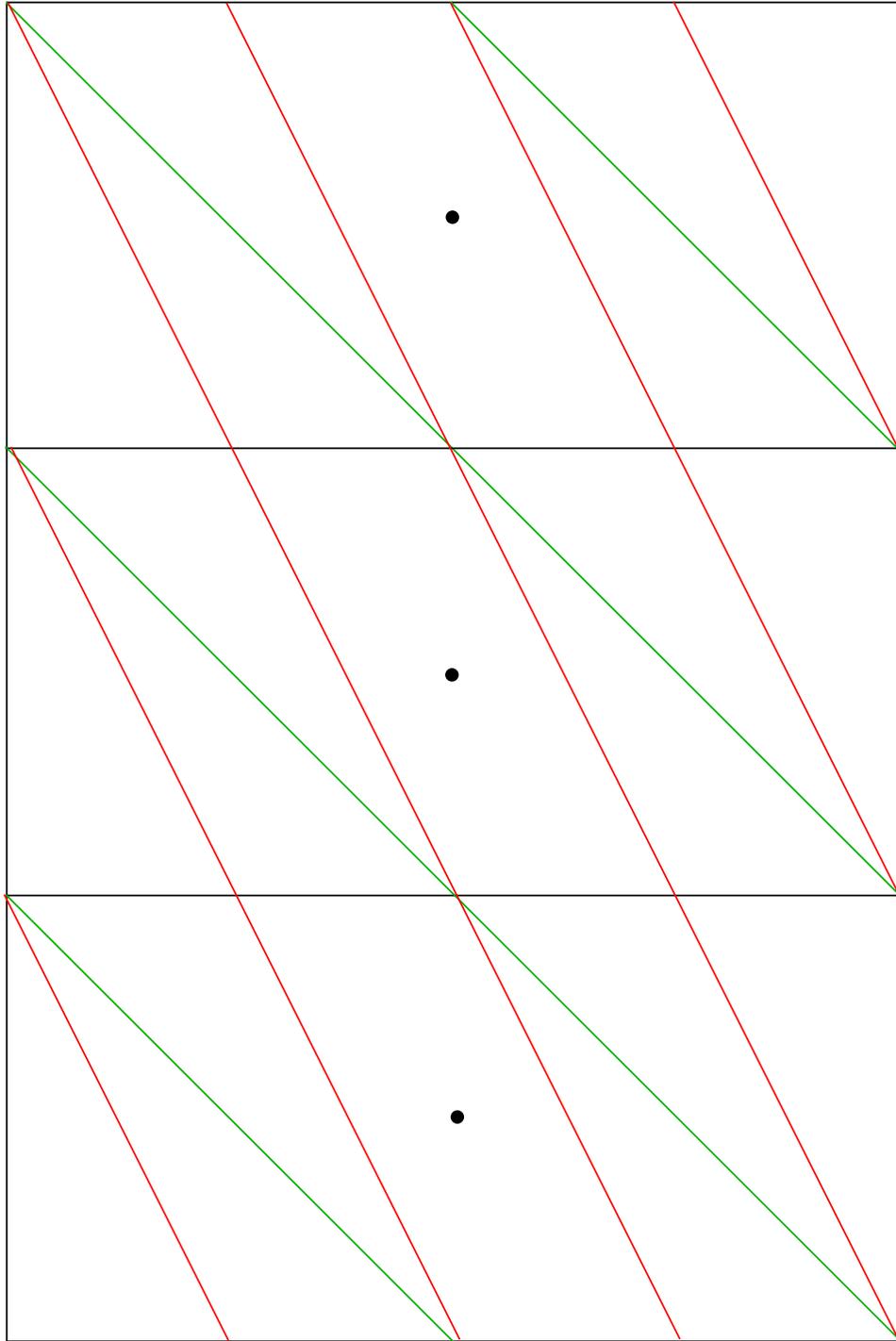


Figure 4-1: The universal cover of $X(I)$.

We can rephrase this proposition as saying that we can introduce a second grading on $HF^*(L(d), L(d+n))$ where $HF^{*,a}(L(d), L(d+n))$ is generated by $q_{a,i}$ for $i \in \{0, 1, \dots, \lfloor \frac{n-|a|}{2} \rfloor\}$, and that μ^2 respects this grading.

Now we show that any triangles contributing to the product of interest are sections of the Lefschetz fibration $\tilde{w} : \tilde{X}(B) \rightarrow \tilde{X}(I)$:

Proposition 4.1.2. *Let $u : S \rightarrow X(B)$ be a pseudo-holomorphic triangle contributing to the component of p in $\mu^2(q_2, q_1)$, where S is the standard disk with three boundary punctures. Then $\tilde{u} : S \rightarrow \tilde{X}(B)$ is a section of \tilde{w} over a triangle T in $\tilde{X}(I)$ bounded by appropriate lifts of $\ell(0), \ell(n)$ and $\ell(n+m)$.*

Moreover, there is a holomorphic isomorphism $\tau : S \rightarrow T$ and a pseudo-holomorphic section $s : T \rightarrow \tilde{X}(B)$ such that $\tilde{u} = s \circ \tau$.

Conversely, any pseudo-holomorphic section $s : T \rightarrow \tilde{X}(B)$ with boundary on $L(0), L(n), L(n+m)$ which maps the corners to q_2, q_1, p contributes to the coefficient of p in $\mu^2(q_2, q_1)$.

Proof. The triangle T and the lifts of the $\ell(d)$ are determined by the considerations from the previous proposition. Clearly $\tilde{w} \circ \tilde{u}$ defines a 2-chain in $\tilde{X}(I)$, which by maximum principle is supported on T . By positivity of intersection with the fibers of \tilde{w} , all components of this 2-chain are positive, and the map $\tilde{w} \circ \tilde{u} : S \rightarrow T$ is a ramified covering. However, if the degree were greater than one, then ∂S would have to wind around $\ell(0), \ell(n), \ell(n+m)$ more than once, contradicting the boundary conditions we placed on the map u .

Since the projection $\tilde{w} : \tilde{X}(B) \rightarrow \tilde{X}(I)$ is holomorphic, the composition $\tilde{w} \circ \tilde{u} : S \rightarrow T$ is a holomorphic map which sends the boundary to the boundary and the punctures to the punctures, so it is a holomorphic isomorphism, and we let τ be its inverse.

For the converse, uniformization for the disk with three boundary punctures yields a unique map $\tau : S \rightarrow T$, and the composition $\tilde{u} = s \circ \tau$ is the desired triangle in $\tilde{X}(B)$. Composing this with the map induced by the covering $\tilde{X}(I) \rightarrow X(I)$ yields the triangle in $X(B)$. \square

Proposition 4.1.3. *Suppose $q_1 = q_{a,i}$ lies in the fiber indexed by a/n , and $q_2 = q_{b,j}$ in the fiber indexed by b/m . Then the sections in Proposition 4.1.2 cover the critical values of the Lefschetz fibration k times, where*

- $k = 0$ if a and b are both nonnegative or both nonpositive.
- $k = \min(|a|, |b|)$ if a and b have different signs.

Proof. We identify $\tilde{X}(I)$ with $[-1, 1] \times \mathbb{R}$. The critical values lie at the points $\{0\} \times (\mathbb{Z} + \frac{1}{2})$.

If a and b are both nonnegative or both nonpositive, then the triangle T is entirely to one side of the vertical line $\{0\} \times \mathbb{R}$ where all the critical values lie.

Suppose a and b have opposite signs and $|a| \leq |b|$. Then the output point lies at $(a+b)/(n+m)$, which has the same sign as b . The side of T corresponding to $\ell(n)$ crosses the line $\{0\} \times \mathbb{R}$ at $(0, a)$, while the side corresponding to $\ell(0)$ crosses at $(0, 0)$, so the distance is $|a|$, and in fact the set $\{0\} \times (\mathbb{Z} + \frac{1}{2})$ contains $|a|$ points in this interval.

If $|b| \leq |a|$, the output point at $(a+b)/(n+m)$ has the same sign as a , and so we need to look at where $\ell(n+m)$ intersects the line $\{0\} \times \mathbb{R}$. This happens at $(0, a+b)$, and the distance to $(0, a)$ is $|b|$. □

With the notation introduced so far, we can state the main result of our computation for $\mu^2(q_{b,j}, q_{a,i})$.

Proposition 4.1.4. *Suppose that $q_{a,i} \in HF^0(L(0), L(n))$ and $q_{b,j} \in HF^0(L(n), L(n+m))$, where the notation is taken from section 3.3.4, and let k be as in Proposition 4.1.3. Then*

$$\mu^2(q_{b,j}, q_{a,i}) = \sum_{s=0}^k \binom{k}{s} q_{a+b, i+j+s} \quad (4.1)$$

Proof. This proposition is the combination of Propositions 4.3.1, 4.4.2, 4.5.2, 4.5.3, and 4.6.1. □

We shall now reformulate Proposition 4.1.4 in algebro-geometric terms. Let

$$A = \bigoplus_{d=0}^{\infty} A_d = \bigoplus_{d=0}^{\infty} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong \mathbb{K}[x, y, z] \quad (4.2)$$

be the homogeneous coordinate ring of \mathbb{P}^2 , where we regard $A_d = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ as the space of degree d homogeneous polynomials in the three variables x, y, z . Define

$$p = xz - y^2, \quad (4.3)$$

and set, for $a \in \{-d, \dots, d\}$, $i \in \{0, \dots, \lfloor \frac{d-|a|}{2} \rfloor\}$,

$$Q_{a,i} = \begin{cases} x^{-a} p^i y^{d+a-2i} & \text{if } a \leq 0 \\ z^a p^i y^{d-a-2i} & \text{if } a > 0 \end{cases} \in A_d. \quad (4.4)$$

Proposition 4.1.5. *Take $Q_{a,i} \in A_n$ and $Q_{b,j} \in A_m$, then in A ,*

$$Q_{a,i} Q_{b,j} = \sum_{s=0}^k \binom{k}{s} Q_{a+b, i+j+s} \quad (4.5)$$

where $k = \min(|a|, |b|)$ if a and b have different signs, and $k = 0$ otherwise.

Proof. The case where a and b have the same sign is obvious.

Suppose that $a \leq 0$ and $b \geq 0$, and suppose that $|a| \leq |b|$. Then we have $a+b \geq 0$, and $k = -a$.

$$Q_{a,i} Q_{b,j} = x^{-a} p^i y^{n+a-2i} z^b p^j y^{m-b-2j} = z^{a+b} (xz)^k p^{i+j} y^{n+m+a-b-2(i+j)} \quad (4.6)$$

Since $xz = p + y^2$, we have

$$(xz)^k = \sum_{s=0}^k \binom{k}{s} p^s y^{2(k-s)} \quad (4.7)$$

$$Q_{a,i} Q_{b,j} = \sum_{s=0}^k \binom{k}{s} z^{a+b} p^{i+j+s} y^{(n+m)-(a+b)-2(i+j+s)} \quad (4.8)$$

Where the monomial on the right is just $Q_{a+b, i+j+s}$.

The other cases are similar. □

To agree with conventions found elsewhere, define the product for Floer cohomol-

ogy as $q_1 \cdot q_2 = (-1)^{|q_1|} \mu^2(q_2, q_1)$. In case all morphisms have degree zero the sign is trivial. The following proposition states how our Lagrangian intersections give rise to a distinguished basis of the homogeneous coordinate ring A .

Proposition 4.1.6. *The map $\psi_{d,n} : HF^0(L(d), L(d+n)) \rightarrow A_n$ defined by*

$$\psi_{d,n} : q_{a,i} \mapsto Q_{a,i} \tag{4.9}$$

is an isomorphism. We have

$$\psi_{d,n+m}(q_1 \cdot q_2) = \psi_{d,n}(q_1) \cdot \psi_{d+n,m}(q_2) \tag{4.10}$$

Proof. That $\psi_{d,n}$ is an isomorphism is because it maps a basis to a basis. The other statement is the combination of Propositions 4.1.4 and 4.1.5. \square

4.2 Extending the fiber

One problem with our Lagrangian boundary conditions $L(d)$ is that they intersect the horizontal boundary $\partial^h X(B)$. This raises the possibility of whether, when a pseudo-holomorphic curve with Lagrangian boundary condition degenerates, any part of it can escape through $\partial^h X(B)$.

We will now describe a technical trick that, by attaching bands to the fiber, allows us to close up the Lagrangians for the purpose of a particular computation, and thereby use only the results in the literature on sections with Lagrangian boundary conditions disjoint from the boundaries of the fibers.

It appears that the way we do this makes no real difference, and in fact almost all of our arguments will concern curves which necessarily remain inside our original cylinder fibers. However, when we try to find the element $c \in HF^*(L, \tau(L))$ appearing in the Floer cohomology exact sequence, we will find a situation where sections actually can escape our original fiber, depending on the particular choice of perturbations.

The starting point for this construction is, given inputs a_1 and a_2 , to consider the portion of the fibration $\tilde{X}(B)|T \rightarrow T$ lying over the triangle T in the base. We recall the assumption from section 3.2 that the symplectic connection is actually flat near the horizontal boundary; after passing to the universal cover of the base, the inner and outer boundary monodromies are no longer a factor, and the fibration is actually symplectically trivial near the horizontal boundary. We also assume that the boundary intersections of our Lagrangians have been positively perturbed as in Remark 7. Hence, after trivializing the fibration near the horizontal boundary, we find that in each fiber of F_z of $\tilde{X}(B)|T \rightarrow T$, there are six points on ∂F_z (three on either component), arising as the parallel translations of the boundary points of $L(0)$, $L(n)$, and $L(n+m)$. These are the points where the Lagrangians $L(0)$, $L(n)$, $L(n+m)$ are allowed to intersect ∂F_z , though note that $L(d) \cap \partial F_z$ is only nonempty if $z \in \ell(d)$ is on the appropriate boundary component of the triangle T . The two sets of three points on each component of ∂F_z are matched according to which Lagrangian they come from, and we extend the fiber F_z to \hat{F}_z by attaching three bands running between the two components of F_z according to this matching. We call the resulting fibration $\hat{X}_T \rightarrow T$. We have an embedding $\iota : \tilde{X}(B)|T \rightarrow \hat{X}_T$.

Over the component of ∂T where the Lagrangian boundary condition $L(0)$ lies, we extend $L(0)$ to $\hat{L}(0)$, closing it up fiberwise to a circle by letting it run through the corresponding band in \hat{F}_z . Similarly we close up $L(n)$ and $L(n+m)$ to their hat-versions.

Figure 4-2 shows the cylinder with a band attached. The actual extended fiber has three such bands.

It is immediate from the construction that $\hat{X}_T \setminus \text{image}(\iota)$ is symplectically a product. Let us use a complex structure which is also a product in this region. Transversality can be achieved using such structures because all intersection points lie in $\text{image}(\iota)$, and hence any pseudo-holomorphic section must also pass through $\text{image}(\iota)$, where we are free to perturb J as usual. We have the following proposition:

Proposition 4.2.1. *Any pseudo-holomorphic section $s : T \rightarrow \hat{X}_T$ with boundary conditions $\hat{L}(d)$, $d = 0, n, n+m$ lies within $\text{image}(\iota)$.*

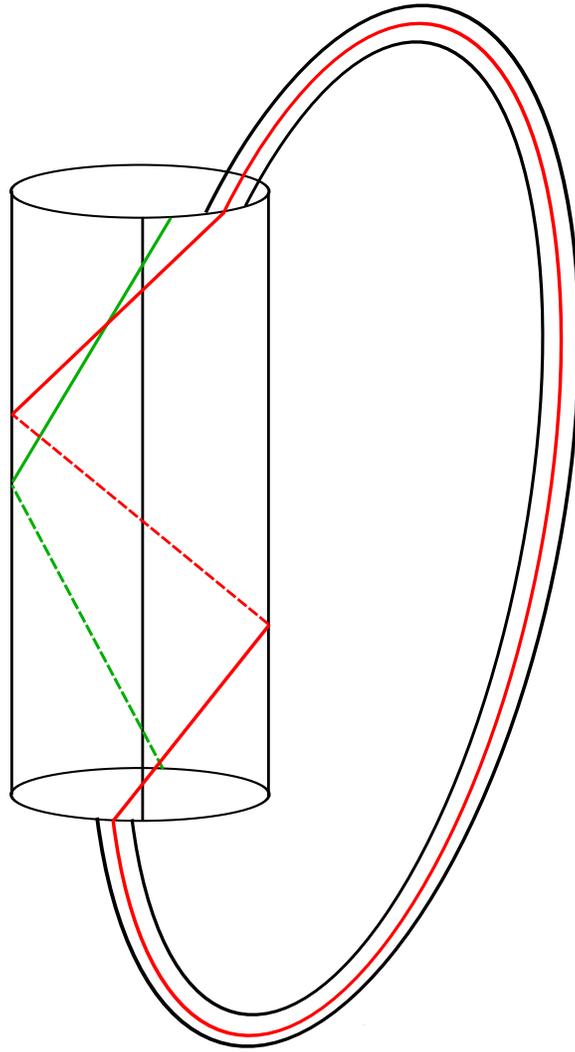


Figure 4-2: Attaching a band to close up one of the Lagrangians in the fiber.

Proof. Suppose not, then by projection to the fiber in $\hat{X}_T \setminus \text{image}(\iota)$, we have a holomorphic curve in one of the bands with boundary on the Lagrangian core of the band. By the maximum principle this map must be constant, but then it does not satisfy the boundary condition. \square

4.3 Degenerating the fibration

By propositions 4.1.2 and 4.2.1, in order to compute the Floer product between two morphisms $q_1 \in HF^0(L(0), L(n))$ and $q_2 \in HF^0(L(n), L(n+m))$, it is just as good to count sections of the fibration $\hat{X}_T \rightarrow T$ with Lagrangian boundary conditions $\hat{L}(0)$, $\hat{L}(n)$, $\hat{L}(n+m)$.

In order to obtain these counts, we will consider a degeneration of the Lefschetz fibration. We consider a one-parameter family of Lefschetz fibrations $\hat{X}^r \rightarrow T^r$, which for $r = 1$ is simply the one we started with. As r goes to zero, the fibration deforms so that all of the k critical values contained within T move toward the side of T corresponding to $\ell(n)$. In the limit, a disc bubble appears around each critical value, and at $r = 0$, the base T has broken into a $(k+3)$ -gon, with k “new” vertices along the side corresponding to $\ell(0)$, each of which joins to a disk with a single critical value. We can equip each fibration with Lagrangian boundary conditions varying continuously with r , and degenerating to a collection of Lagrangian boundary conditions for each of the component fibrations at $r = 0$.

The base of the Lefschetz fibration undergoes the degeneration shown in figure 4-3. This figure shows specifically the case for the product of $x \in CF^*(L(0), L(1))$ and $z \in CF^*(L(1), L(2))$. The marked point on the upper portion is the Lefschetz critical value, while the marked points on the lower portion are the Lefschetz critical value and a node.

In order to perform this construction carefully, it is better describe this family as a smoothing of the degenerate $r = 0$ end. Let T^0 be a disk with $(k+3)$ boundary punctures, and note that the conformal structure of T^0 is fixed. Let $\hat{X}^0 \rightarrow T^0$ be a symplectically trivial Lefschetz fibration. Let D_1, \dots, D_k denote the k disks with one

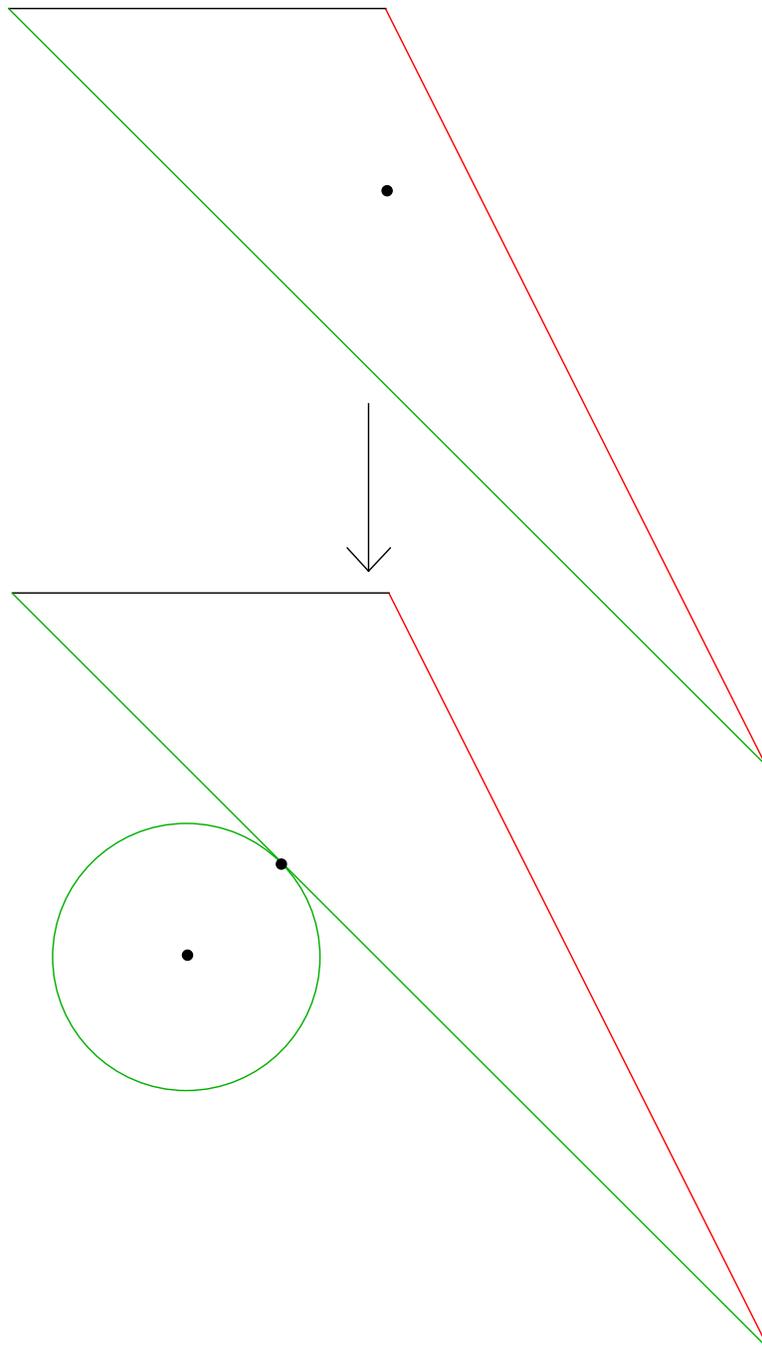


Figure 4-3: Degenerating the fibration.

boundary puncture that are our “bubbles”. For each i let $E_i \rightarrow D_i$ denote a Lefschetz fibration with a single critical value, and which is trivial near the puncture. Let the monodromy around ∂D_i be denoted τ_i .

We will equip each of the components of this fibration with a Lagrangian boundary condition.

- The base T^0 has one boundary component corresponding to $\ell(0)$, one boundary component corresponding to $\ell(n+m)$, and $(k+1)$ boundary components corresponding to $\ell(n)$. Since the fibration $\hat{X}^0 \rightarrow T^0$ is trivial, it suffices to describe each boundary condition in the fiber. Over the point where $\ell(0)$ and $\ell(n+m)$ come together, we identify the fiber with that of $\hat{X}^1 \rightarrow T^1$, and take $\hat{L}(0)^0$ and $\hat{L}(n+m)^0$ to be the corresponding Lagrangians.
- Over the k boundary components of T^0 corresponding to $\ell(n)$, we construct a sequence $\hat{L}(n)_i^0$ of Lagrangians. At the puncture where $\ell(0)$ and $\ell(n)$ come together, we take $\hat{L}(n)_0^0$ to have the same position relative to $\hat{L}(0)^0$ (already constructed) that $\hat{L}(n)$ has relative to $\hat{L}(0)$ in the original fibration. As we pass each of the new punctures where the disks are attached, the monodromies τ_i must be applied. So we let

$$\hat{L}(n)_i^0 = \tau_i(\hat{L}(n)_{i-1}^0) \tag{4.11}$$

This can be done so that, over the puncture where $\ell(n)$ and $\ell(n+m)$ come together, $\hat{L}(n)_k^0$ and $\hat{L}(n+m)^0$ intersect as the original $\hat{L}(n)$ and $\hat{L}(n+m)$ do.

- Over the k disks D_i , we take a Lagrangian boundary condition which interpolates between $\hat{L}(n)_{i-1}^0$ and $\hat{L}(n)_i^0$. This is tautological by equation (4.11).

Note that at this stage we only care about the twists τ_i up to Hamiltonian isotopy, but we will make a particular choice later on, which is related to the “nonnegative curvature” condition for Lefschetz fibrations. This will give us τ_i such that $\hat{L}(n)_{i-1}^0$ and $\hat{L}(n)_i^0$ have one intersection point over i -th new puncture of T^0 .

Since the boundary conditions agree over the corresponding punctures, we can glue the components over T^0 and D_1, \dots, D_k with large gluing length in order to obtain $\hat{X}^\epsilon \rightarrow T^\epsilon$ for small $\epsilon > 0$, which has three boundary components and k critical values all near the $\ell(n)$ boundary. Clearly there is a family of Lefschetz fibrations interpolating between $\hat{X}^\epsilon \rightarrow T^\epsilon$ back to our original $\hat{X}^1 \rightarrow T^1$, along which the critical values move back to their original points.

As far as pseudo-holomorphic sections are concerned, as long as the gluing length is large, the sections for $r = \epsilon$ will be obtained from sections over T^0 and D_1, \dots, D_k by gluing sections with matching values at the punctures. During the deformation from $r = \epsilon$ to $r = 1$, no Floer strip breaking can occur because our Lagrangians $\hat{L}(d)$, $d = 0, n, n + m$ do not bound any strips, even topologically. Hence we have the following:

Proposition 4.3.1. *The count of pseudo-holomorphic sections of $\hat{X}^1 \rightarrow T^1$, with Lagrangian boundary conditions $\hat{L}(d)$, $d = 0, n, n + m$ is obtained from the counts of sections of $\hat{X}^0 \rightarrow T^0$ and $E_i \rightarrow D_i$ by gluing together sections whose values over the punctures match.*

4.4 Horizontal sections over a disk with one critical value

It is possible to determine the element $c \in CF^0(L, \tau(L))$ using the techniques of horizontal sections developed by Seidel in [32], in particular we will apply results from section 2.5 of that paper, so we adopt its notation.

In order for this technique to work, we need to set up a model Lefschetz fibration over a base S , the disk with one end, carefully in order to ensure that all the sections we need to count are in fact horizontal. The key properties are

- Transversality of the boundary conditions over the strip-like end. This means that we cannot use a standard model Dehn twist fibration, but need to introduce a perturbation somewhere.

- Nonnegative curvature. The standard model Dehn twist fibration has nonnegative curvature, but requiring the perturbation to have nonnegative curvature imposes a constraint.
- Vanishing action of horizontal sections. This imposes a further constraint on the perturbation.

As we progress through the construction, the definitions of all of these terms will be recalled.

The first step is to construct a fibration which is flat away from the critical point, following section 3.3 of [32]. Let S be the Riemann surface $\{\operatorname{Re} z \leq 0, |\operatorname{Im} z| \leq 1\} \cup \{|z| \leq 1\}$, so it is a negative half-strip which has been rounded off with a half-disk.

Let M denote the fiber, which is a cylinder with the vanishing cycle V in it, onto which several bands have been attached. The circle running through the core of one of these bands is L . Equip M with a symplectic form $\omega = d\theta$. Over $S^- = (-\infty, -1] \times [-1, 1]$, we let $\pi : E^- \rightarrow S^-$ be a trivial fibration and equip E with 1-form Θ and 2-form $\Omega = d\Theta$ which are pulled back from the fiber (to get a symplectic form we add the pullback a positive 2-form $\nu \in \Omega^2(S)$, but this does not affect the symplectic connection). Following the pasting procedure described in section 3.3 of [32] (though with the cut on the right rather than the left), we can complete this fibration $\pi : E \rightarrow S$ to one where the monodromy around the boundary is τ_V , a standard model Dehn twist supported near the vanishing cycle. This has nonnegative curvature, is actually trivial on the part of the fiber away from the support of the Dehn twist, and is flat over the part of the base away from the critical value.

For this fibration, there is a Lagrangian boundary condition which over the end corresponds to the pair $(L, \tau_V(L))$. Of course, these are not transverse since τ_V is identity in the band. Hence we will perturb the symplectic form by a term which depends on a Hamiltonian. We will introduce the perturbation in a neighborhood of the edge $(-\infty, -1] \times \{1\}$. Let β be a cutoff function which

- is supported in $U = \{s + it \mid -2 \leq s \leq -1, 1 - \epsilon \leq t \leq 1\}$,

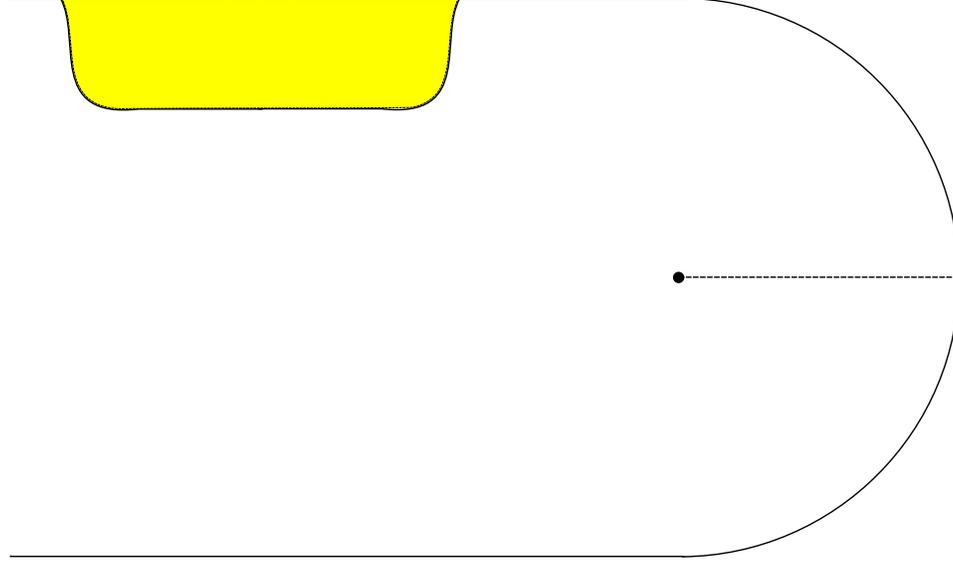


Figure 4-4: The base of the fibration with the region of perturbation shaded.

- vanishes along the bottom, left and right sides of U , and
- has $\partial\beta/\partial t \geq 0$.

Figure 4-4 shows the base of the fibration, with the region where β has support shaded.

Let H be a Hamiltonian function on the fiber M , and let X_H be its vector field defined by $\omega(\cdot, X_H) = dH$. Then over S^- , where the fibration was originally trivial with $\Omega = d\Theta$ pulled back from the fiber, consider

$$\Theta' = \Theta + H\beta ds \tag{4.12}$$

$$\Omega' = d\Theta' = \Omega + \beta dH \wedge ds - H(\partial\beta/\partial t) ds \wedge dt \tag{4.13}$$

This modifies the symplectic connection as follows: Let $Y \in TE^v$ denote the general vertical vector. Then if $Z \in TS$, and Z^h is its horizontal lift

$$0 = \Omega'(Y, Z^h) = \Omega(Y, Z^h) + \beta dH(Y) ds(Z) \tag{4.14}$$

If we denote by Z again the horizontal lift with respect to the trivial connection, we have $Z^h = Z - \beta ds(Z)X_H$. Since $\beta \geq 0$, this means that as we parallel transport in the *negative* s -direction through the region U , we pick up a bit of the Hamiltonian flow of H , compared with the trivial connection. By adjusting the function β , we can ensure that the parallel transport along the boundary in the positive sense picks up ϕ_H , the time 1 flow of H .

We must compute the curvature of this connection. This is the 2-form on the base with values in functions on the fibers given by $\Omega'(Z_1^h, Z_2^h)$. It will suffice to compute for $Z_1 = \partial/\partial s$ and $Z_2 = \partial/\partial t$, a positive basis. We have $Z_1^h = \partial/\partial s - \beta X_H$ and $Z_2 = \partial/\partial t$.

$$\begin{aligned} \Omega'(Z_1^h, Z_2^h) = \\ \Omega(Z_1^h, Z_2^h) + \beta dH(Z_1^h)ds(Z_2^h) - \beta dH(Z_2^h)ds(Z_1^h) - H(\partial\beta/\partial t)ds \wedge dt(Z_1^h, Z_2^h) \end{aligned} \quad (4.15)$$

The first term vanishes because Z_2^h is horizontal for the trivial connection. The second term vanishes because $ds(Z_2^h) = 0$. The third because $dH(Z_2^h) = 0$. This leaves $-H(\partial\beta/\partial t)ds \wedge dt(Z_1^h, Z_2^h) = -H(\partial\beta/\partial t)$. By our assumptions on β , this is nonnegative as long as $H \leq 0$.

We equip the deformed fibration with a Lagrangian boundary condition Q given by parallel transport of L around the boundary. This picks up a Dehn twist and the time 1 flow of H , so over the end we have the pair $(L, \phi_H(\tau_V(L)))$.

A *horizontal section* is a map $u : S \rightarrow E$ such that $Du(TS) = TE^h$. The importance of such sections is that, while they are determined by the symplectic connection, they are pseudo-holomorphic for any *horizontal complex structure* J , which is an almost complex structure that preserves TE^h .

The *action* $A(u)$ of a section u is defined to be $\int_S u^*\Omega'$. The symplectic area of u is then $A(u) + \int_S \nu$. The identity relating action to energy is ([32], eq. 2.10)

$$\frac{1}{2} \int_S \|(Du)^v\|^2 + \int_S f(u)\nu = A(u) + \int_S \|\bar{\partial}_J u\|^2 \quad (4.16)$$

for any horizontal complex structure J . Here $Du = (Du)^h + (Du)^v$ is the splitting induced by the connection, and f is the function determined by the curvature as $f(\pi^*\nu|TE^h) = \Omega|TE^h$. In our example f is supported near the critical point and in $\text{supp } \beta$, where $f = -H(\partial\beta/\partial t)$.

A direct consequence of (4.16) is the following:

Proposition 4.4.1. *If the curvature of $\pi : E \rightarrow S$ is nonnegative, and if u is a J -holomorphic section with $A(u) = 0$, then u is horizontal and the curvature function f vanishes on the image of u .*

With all this in mind, we will choose our Hamiltonian $H : M \rightarrow \mathbb{R}$ as follows:

- $H \leq 0$ and $H = 0$ near ∂M . This ensures that the fibration is still trivial near the horizontal boundary and that the curvature is nonnegative within $\text{supp } \beta$.
- H achieves its global maximum of 0 near ∂M and on the “cocore” of the band in M . It achieves its minimum along the vanishing cycle, and has no other critical points in the cylinder or lying on L (which intersects the vanishing cycle and the cocore once). The first condition implies that horizontal sections passing through the cocore do not pick up any curvature, while the second is there in order to ensure that $(L, \phi_H(\tau_V(L)))$ is a transverse pair.

In fact, if the function H is chosen appropriately, then $L \cap \phi_H(\tau_V(L))$ will consist of one point x lying on the cocore of the band, which has degree zero (giving L some grading and $\phi_H(\tau_V(L))$ the induced grading).

After this setup, we come to the problem of determining the set $\mathcal{M}_J(x)$ of J -holomorphic sections $u : S \rightarrow E$ which are asymptotic to $x \in L \cap \phi_H(\tau_V(L))$ over the end.

Proposition 4.4.2. *Let J be a horizontal complex structure. Then $\mathcal{M}_J(x)$ consists of precisely one section. It is horizontal, has $A(u) = 0$, and is regular.*

Proof. The first step is to construct a horizontal section. Any horizontal section, if it exists, is determined by parallel transport of the point $x \in L \cap \phi_H(\tau_V(L))$ throughout

E . Consider the section over S^- given by $u^- : (s, t) \mapsto (s, t, x)$. This is clearly horizontal outside $\text{supp } \beta$, since the fibration is trivial over $S^- \setminus \text{supp } \beta$. In $\text{supp } \beta$, $(TE^h)_{(s,t,x)}$ is the same as for the trivial connection, because x lies at a critical point of H , so the section is horizontal there as well. Near the singularity, the fibration is trivial in the band where x lies, so u^- extends to a horizontal section $u : S \rightarrow E$.

For this section, we compute $A(u) = \int_S u^* \Omega'$. Since u lies in the region where τ_V is trivial, $\int_{S \setminus \text{supp } \beta} u^* \Omega' = 0$. Since u passes through the point x where $H(x) = 0$, the contribution $\int_{\text{supp } \beta} u^* \Omega' = \int_{\text{supp } \beta} -H(x)(\partial\beta/\partial t)\nu$ vanishes.

Since $A(u) = 0$, any $u' \in \mathcal{M}_J(X)$ must also have action 0. Because the curvature of $\pi : E \rightarrow S$ is nonnegative, proposition 4.4.1 implies that u' is horizontal. Since u' and u are both asymptotic to x , they are equal. Hence $\mathcal{M}_J(x) = \{u\}$.

It remains to show that u is regular. The linearization of parallel transport along u trivializes $u^*(TE^v)$ such that the boundary condition $u^*(TQ)$ is mapped to a family of Lagrangian subspaces which, as we traverse ∂S in the positive sense, tilt clockwise by a small amount. Hence $\text{ind } D_{u,J} = 0$. On the other hand, Lemma 2.27 of [32] applies to the section u , implying $\ker D_{u,J} = 0$. Hence $\text{coker } D_{u,J} = 0$ as well. \square

4.5 Polygons with fixed conformal structure

Recall from section 4.3 the trivial fibration $\pi : \hat{X}^0 \rightarrow T^0$, with fiber M , where T^0 is a disk with $(k+3)$ boundary punctures. We have $\hat{X}^0 = M \times T^0$ symplectically. We equip \hat{X}^0 with a product almost complex structure $J = J_M \times j$, where j is the complex structure on T^0 .

Proposition 4.5.1. *The (j, J) -holomorphic sections $u : T^0 \rightarrow \hat{X}^0$ are in one-to-one correspondence with (j, J_M) -holomorphic maps $u_M : T^0 \rightarrow M$.*

Proof. Given $u : T^0 \rightarrow \hat{X}^0$, write $u = (u_M, u_{T^0})$ with respect to the product splitting. Since J is a product each component is pseudo-holomorphic in the appropriate sense. But u_{T^0} is the identity map because u is a section. This correspondence is clearly invertible. \square

This reduces the problem of counting sections $u : T^0 \rightarrow \hat{X}^0$ to the problem of counting maps $u_M : T^0 \rightarrow M$. We emphasize that (T^0, j) is a Riemann surface with a fixed conformal structure.

The maps $u_M : T \rightarrow M$ are holomorphic maps between Riemann surfaces, and hence their classification is mostly combinatorial. However, because we are in a situation where the conformal structure on the domain is fixed, we are not quite in the situation described, for example, in Section 13 of [34].

The holomorphic curves we are looking for have non-convex corners and hence boundary branch points or “slits,” and if the situation is complicated enough they may also have branch points in the interior. However, the condition that the conformal structure of the domain is fixed makes this an index zero problem, which is to say it prevents these slits and branch points from deforming continuously. The question is then, given a combinatorial type of such a curve, what conformal structures (with multiplicity) can be realized by holomorphic representatives?

The $(k + 3)$ boundary components of T^0 are equipped with Lagrangian boundary conditions $\hat{L}(0)^0, \hat{L}(n)_0^0, \dots, \hat{L}(n)_k^0, \hat{L}(n + m)^0$. Recall that over the ends where disk bubbles are attached we have $\hat{L}(n)_i^0 = \tau_i(\hat{L}(n)_{i-1}^0)$, where τ_i is the monodromy around the i -critical point inside the i -th disk bubble. In section 4.4, we refined the construction and made a particular choice for this monodromy:

$$\tau_i = \phi_H \circ \tau_V \tag{4.17}$$

Since all of these symplectomorphisms are isotopic, we will denote them all by τ for most of this section.

In order to simplify notation, for the rest of this section 4.5 we will drop the hats and superscript zeros and denote by

$$L(0), L(n), \tau L(n), \tau^2 L(n), \dots, \tau^k L(n), L(n + m) \subset M \tag{4.18}$$

the Lagrangians in the fiber M which give rise to the boundary conditions over the $(k + 3)$ -punctured disk T . Though the monodromies are all denoted by τ , it is

good to actually choose the perturbations slightly differently so as to ensure that this collection of Lagrangians is in general position in M ; this is necessary for the argument in Lemma 4.5.9.

Recall that $q_1 = q_{a,i}$ and $q_2 = q_{b,j}$ are the morphisms whose product we wish to compute. We now regard $q_{a,i} \in L(0) \cap L(n)$ and $q_{b,j} \in \tau^k L(n) \cap L(n+m)$. Let $x_i \in \tau^{i-1} L(n) \cap \tau^i L(n)$ denote the unique intersection point.

Recall that the possible output points $q_{a+b,h} \in L(0) \cap L(n+m)$ are indexed by $h \in \{0, 1, \dots, \lfloor \frac{(n+m)-|a+b|}{2} \rfloor\}$.

We can now state the main results of this section.

Proposition 4.5.2. *If h is such that $0 \leq h - (i + j) \leq k$, then there are $\binom{k}{h-(i+j)}$ homotopy classes of maps $u : T \rightarrow M$ satisfying the boundary conditions and asymptotic to $q_{a,i}, x_1, \dots, x_k, q_{b,j}, q_{a+b,h}$ at the punctures. For h outside this range the set of such homotopy classes is empty.*

Proposition 4.5.3. *Furthermore, for each of these homotopy classes, and for each complex structure j on T , there is exactly one holomorphic representative $u : T \rightarrow M$.*

The strategy of proof is first to prove Proposition 4.5.2. Then we show the existence of holomorphic representatives for some conformal structure, and show that the number of representatives does not depend on the conformal structure. By degenerating the domain we are able to show uniqueness.

4.5.1 Homotopy classes

The analysis of homotopy classes requires some explicit combinatorics, which we shall now set up. Recall that M is a cylinder with three bands attached, one for each of $L(0), L(n), L(n+m)$. We will classify homotopy classes of maps $u : T \rightarrow M$ by their boundary loop ∂u , which must of course be contractible, traverse $L(0), L(n), \dots, \tau^k L(n), L(n+m)$ in order, and hit the intersection points $q_{a,i}, x_1, \dots, x_k, q_{b,j}, q_{a+b,h}$ in order.

Let us use $L(n)$ to frame the cylinder, so that winding around the cylinder is computed with respect to $L(n)$. Let $x \in M$ be a basepoint which is located in the

band near the intersection points x_r . Let $\alpha \in \pi_1(M, x)$ denote a loop that enters the cylinder from the bottom, veers right, winds around once, and goes back to x . We also have a class $\beta \in \pi_1(M)$ that is represented by the loop $L(n)$, oriented upward through the cylinder. Note that α and β generate a free group in $\pi_1(M)$.

With these conventions, $L(0)$ winds $-(n - |a|)/2$ times, $\tau^r L(n)$ winds r times, and $L(n + m)$ winds $(m - |b|)/2 + k$ times, within the cylinder.

The only unknown is how many times the boundary path traverses $L(0)$, $L(n)$, $\tau L(n)$, etc., as we traverse the boundary in the positive sense. In order for the whole loop to be contractible, it cannot pass through the bands corresponding to $L(0)$ and $L(n + m)$. This is because $\tau^r L(n)$, $0 \leq r \leq k$ can only contribute words in α and β , which can never cancel loops through the other bands. Hence the portion of our loop along $L(0)$ and $L(n + m)$ lies within the cylinder, and therefore it is determined by the choices of $q_{a,i}$, $q_{b,j}$ and $q_{a+b,h}$.

As for the portion of the loop along $\tau^r L(n)$, this can be represented by a sequence of integers $(\delta_r)_{r=0}^k$, where δ_r represents the number of times we wind around $\tau^r L(n)$.

With these conventions in place, we can compute the winding of a choice of paths satisfying the boundary and asymptotic conditions. We record the parts of the computation:

- Passing from x_k to $q_{b,j}$ by a short upward path on $\tau^k L(n)$ contributes

$$\left(1 - \frac{j}{(m - |b|)/2}\right) (k) \quad (4.19)$$

- The winding around the cylinder along $L(n + m)$ as we pass from $q_{b,j}$ to $q_{a+b,h}$ is

$$\left[\frac{h}{(n + m - |a + b|)/2} - \frac{j}{(m - |b|)/2}\right] (-1)^{((m - |b|)/2 + k)} \quad (4.20)$$

- The winding around the cylinder along $L(0)$ as we pass from $q_{a+b,h}$ to $q_{a,i}$ is

$$\left[\frac{i}{(n - |a|)/2} - \frac{h}{(n + m - |a + b|)/2}\right] (n - |a|)/2 \quad (4.21)$$

- Passing from $q_{a,i}$ to x_1 by a short downward path on $L(n)$ contributes 0 to the winding around the cylinder.

Adding up these contributions and using the fact that $(m-|b|)/2+(n-|a|)/2+k = (n+m-|a+b|)/2$, we get a total of $i+j-h+k$. Thus, if we go up on $\tau^k L(n)$ and down on $L(n)$, we pick up the class $\alpha^{i+j-h+k} \in \pi_1(M, x)$ for the loop from x_k to x_1 .

Thus, the class $\alpha^{i+j-h+k}$ corresponds to the choice $\delta_r = 0$ for $0 \leq r \leq k$. The homotopy class of any other path can be computed from this as follows:

- Taking another path on $L(n)$ contributes a factor β^{δ_0} on the right.
- Passing from x_r to x_{r+1} along $\tau^r L(n)$ contributes the word

$$(\alpha^r \beta)^{\delta_r} \tag{4.22}$$

where $\delta_r \in \mathbb{Z}$, and this class is added on the right.

- Going down on $\tau^k L(n)$ rather than up contributes the class $(\alpha^k \beta)^{\delta_k}$ on the left, which up to conjugation is the same as adding the class $(\alpha^k \beta)^{\delta_k}$ on the right.

Thus the class in question is

$$\alpha^{i+j-h+k} \prod_{r=0}^k (\alpha^r \beta)^{\delta_r} \tag{4.23}$$

The key condition is that this class is trivial in $\pi_1(M)$. This means in particular that all of the β 's must cancel out. Because α and β generate a free group we have the following:

Lemma 4.5.4. *In the word (4.23), the β 's cancel out if and only if $\delta_r \in \{-1, 0, 1\}$ for $0 \leq r \leq k$, the first nonzero δ is 1, the last nonzero δ is -1 , and the nonzero δ 's alternate in sign.*

Proof. The first thing to see is that $|\delta_r| \leq 1$ for $1 \leq r \leq k$. This is because $(\alpha^r \beta)^2 = \alpha^r \beta \alpha^r \beta$ contains an isolated β , while $(\alpha^r \beta)^{-2}$ contains an isolated β^{-1} . Then we can

see that the nonzero δ 's must alternate sign, since having two consecutive δ 's equal to 1 yields $\alpha^{r_1}\beta\alpha^{r_2}\beta$, which has an isolated beta, while having two consecutive δ 's equal to -1 would yield an isolated β^{-1} .

Since the nonzero δ 's in the range $1 \leq r \leq k$ alternate in sign, all the β 's coming from the range $1 \leq r \leq k$ cancel, except for possibly the first or the last. This implies that $|\delta_0| \leq 1$ as well, since the only thing that can cancel β^{δ_0} is the first nonidentity factor or the last nonidentity factor.

Now the β from the first nonidentity factor can only cancel the β from the last nonidentity factor if all the α 's as well as β 's in between cancel. This means that

$$\sum_{r=a}^b r\delta_r = 0 \tag{4.24}$$

for the appropriate range of r : $a \leq r \leq b$. Since the δ 's are in $\{-1, 0, 1\}$ and they alternate in sign, the only solution to this equation is when all $\delta = 0$. In this case, the first and the last nonidentity factors are in fact consecutive, the first has $\delta = 1$, while the last has $\delta = -1$. This shows that it is impossible to have β^{-1} from the first factor cancel a β from the last factor.

In general, we find that each β is cancelled by a β^{-1} from the next nonidentity factor, so that the nonzero δ 's alternate sign for $0 \leq r \leq k$, the first nonzero δ is 1, and the last nonzero δ is -1 . □

By passing to $H_1(M; \mathbb{Z})$, we obtain the relations

$$\sum_{r=0}^k \delta_r = 0 \tag{4.25}$$

$$i + j - h + k + \sum_{r=0}^k r\delta_r = 0 \tag{4.26}$$

Equation (4.25) is implied by Lemma 4.5.4, while (4.26) determines which h the homotopy class contributes to.

The sequences $(\delta_r)_{r=0}^k$ which solve the constraints are in one-to-one correspondence

with sequences $(s_r)_{r=0}^{k-1}$ such that $s_r \in \{0, 1\}$. In one direction, we extend the sequence by $s_{-1} = 0 = s_k$, and set

$$\delta_r = s_r - s_{r-1} \tag{4.27}$$

In the other direction, any sequence δ_r can be integrated to a sequence s_r with $s_{-1} = 0$. Since the signs of the nonzero δ_r alternate, and the first nonzero term is 1, we will have $s_r \in \{0, 1\}$, and since the final nonzero term is -1 , we will have $s_k = 0$, thus inverting the correspondence. This yields 2^k solutions.

Plugging this into equation the summation in (4.26), we have

$$\sum_{r=0}^k r\delta_r = \sum_{r=0}^k r(s_r - s_{r-1}) = \sum_{r=0}^{k-1} (-1)s_r \tag{4.28}$$

because the summation telescopes. This is simply minus the number of 1's in the sequence s_r . Thus we obtain

$$h - (i + j) = k - \sum_{r=0}^{k-1} s_r \tag{4.29}$$

The right hand side is always an integer between 0 and k , and it takes the value s for $\binom{k}{s}$ choices of the sequence $(s_r)_{r=0}^{k-1}$. Thus homotopy classes of maps exist for h such that $0 \leq h - (i + j) \leq k$, and there are $\binom{k}{h-(i+j)}$ such classes. This proves Proposition 4.5.2.

4.5.2 Existence of holomorphic representatives for some conformal structure

The first step in characterizing the holomorphic representatives of these homotopy classes is to prove the existence of holomorphic sections for some conformal structure. This is also essentially combinatorial.

We begin with some general concepts that will be useful in the proof.

Definition 2. Let $\gamma : S^1 \rightarrow \mathbb{C}$ be a piecewise smooth loop. A *subloop* γ' of γ is the restriction $\gamma' = \gamma|_{\bigcup_{\alpha} I_{\alpha}}$ to a collection of intervals $\bigcup_{\alpha} I_{\alpha}$. The indexing set inherits

a cyclic order from S^1 , and we require that for each α , $\gamma(\max I_\alpha) = \gamma(\min I_{\alpha+1})$ is an self-intersection of γ . Thus γ' simply “skips” the portion of γ between $\max I_\alpha$ and $\min I_{\alpha+1}$.

Note that a subloop is not the same as a loop formed by segments of γ joining self-intersections. Such an object is only a subloop if the segments appear in a cyclic order compatible with γ .

Definition 3. A piecewise smooth loop $\gamma : S^1 \rightarrow \mathbb{C}$ is said to have the *(weak) positive winding property* (PWP) if the winding number of γ around any point in $\mathbb{C} \setminus \text{image}(\gamma)$ is nonnegative. The loop γ is said to have the *strong positive winding property* (SPWP) if every subloop $\gamma' \subset \gamma$ has the positive winding property.

Lemma 4.5.5. *A loop $\gamma : S^1 \rightarrow \mathbb{C}$ has SPWP if and only if every simple subloop has PWP.*

Proof. The “only if” direction is contained in the definition. Suppose that every simple subloop of γ has PWP. If γ' is a subloop that is not simple, then by splitting γ' at a self-intersection, we can write γ' as the composition of two proper subloops. Repeating this inductively, we can write γ' as the composition of simple subloops. By hypothesis, each of these subloops winds positively, and the winding of γ' about a point is the sum of the contributions from each of the simple subloops. \square

Lemma 4.5.6. *The strong positive winding property is stable under branched covers in the following sense. Suppose γ has SPWP. Let $y \in \mathbb{C} \setminus \text{image}(\gamma)$ be a point where the winding number of γ around y is $m > 1$. Taking the $m : 1$ branched cover at y , we find that the preimage of γ consists of m closed loops, each of which covers γ once. Let $\tilde{\gamma}$ be one such lift. Then $\tilde{\gamma}$ has SPWP.*

Proof. Suppose that $\tilde{\gamma}$ does not have SPWP. Then some subloop $\tilde{\gamma}'$ does not have PWP. By Lemma 4.5.5, we may take $\tilde{\gamma}'$ to be a *simple* subloop. Thus $\tilde{\gamma}'$ winds around some region once clockwise, and we have $\tilde{\gamma}' = \partial\tilde{C}$, where \tilde{C} is the chain consisting of this region with coefficient -1 . Pushing $\tilde{\gamma}'$ and \tilde{C} forward under the branched cover, we obtain a subloop $\gamma' \subset \gamma$, and chain C such that $\gamma' = \partial C$. Since \tilde{C} is purely

negative, no cancellation can occur when we push forward, and C is purely negative as well. Thus γ' winds negatively about a point in the support of C , which contradicts SPWP for γ . \square

Lemma 4.5.7. *Suppose $\gamma : S^1 \rightarrow \mathbb{C}$ is a piecewise smooth loop with SPWP. Then there is a holomorphic map $u : D^2 \rightarrow \mathbb{C}$ such that $\partial[u] = \gamma$.*

Proof. The idea of the proof is to iteratively take branched covers of the plane and lift γ so as to reduce the density of winding. So let $y \in \mathbb{C} \setminus \text{image}(\gamma)$ be a point where the winding number is $m > 1$. Taking the $m : 1$ branched cover at y , we obtain as in Lemma 4.5.6 a lift $\tilde{\gamma}$ that covers γ once and has SPWP. Repeating this process and using Lemma 4.5.6 to guarantee that the lift always has SPWP, eventually we obtain a *simple* piecewise smooth loop with positive winding. The Riemann mapping theorem yields a map $\tilde{u} : D^2 \rightarrow M$, where M is the Riemann surface resulting from the branched covering construction. Composing \tilde{u} with the covering $M \rightarrow \mathbb{C}$ yields the desired map u . \square

Fix a choice of homotopy class ϕ of polygons $u : T \rightarrow M$, which essentially means fixing a choice for the sequence $(\delta_r)_{r=0}^k$. Passing to the universal cover \tilde{M} of the fiber, fix a choice of lift $\tilde{u} : T \rightarrow \tilde{M}$. Let $y \in \tilde{M}$ be any point. Because the boundary loop $\partial[u]$ is contractible in M , it lifts to a closed loop $\partial[\tilde{u}]$ in \tilde{M} . In fact $\partial[\tilde{u}]$ is contained in a domain which is isomorphic to a domain in \mathbb{C} , and with this identification, we have the following.

Lemma 4.5.8. *The boundary loop $\partial[\tilde{u}]$ has SPWP.*

Proof. The key observation is that the slopes of the paths $L(0), L(n), \dots, \tau^k L(n), L(n+m)$ through the cylinder are positive and monotonically decreasing. If we frame the cylinder using $L(0)$, then

- $L(0)$ has slope ∞
- $L(n)$ has slope $[(n - |a|)/2]^{-1}$
- $\tau^r L(n)$ has slope $[r + (n - |a|)/2]^{-1}$

- $L(n + m)$ has slope $[(n + m - |a + b|)/2]^{-1}$.

Only at the intersection between $L(n + m)$ and $L(0)$ does the slope increase.

Hence as we traverse $\partial[\tilde{u}]$, or any subloop thereof, the slope can only increase at one point. This point is either where the subloop either uses or skips over $L(0)$.

Now we use some elementary plane geometry. Suppose that P is an oriented polygonal path in the plane (possibly self-intersecting), all of whose sides $(S_i)_{i=1}^N$ have positive slope. Suppose that P winds negatively around some point y . We claim the slope has to increase at no fewer than two vertices.

Checking cases proves the claim when $N = 3$ and P is a triangle.

Next we prove the claim if P is a simple N -gon. Suppose for induction that the claim is true for $N < N_0$. If we remove a side S_i from the N_0 -gon P and extend the two incident sides S_{i-1} and S_{i+1} in order to obtain an N -gon P' , then if P has at most one vertex where the slope increases, so does P' , since if the slope decreased at both $S_{i-1}S_i$ and at S_iS_{i+1} , it will decrease at the new vertex $S_{i-1}S_{i+1}$. Since any side can be removed, we can clearly choose S_i so that P' still winds negatively around y . This contradicts the induction hypothesis.

Now suppose P is self-intersecting and winds negatively around y . Using Lemma 4.5.5, we can find a simple subloop P' winding negatively around y . For each vertex v of P' where the slope of P' increases, there is a vertex in the original polygon P where the slope increases, either at v itself, or at some point in the interval of P that was deleted at v . Thus, since P' has at least two slope increases, so does P .

□

Having chosen a lift $\partial[\tilde{u}]$ of the boundary loop, define a 2-chain C on \tilde{M} whose multiplicity at y is the winding number of $\partial[\tilde{u}]$ around y . This has $\partial C = \partial[\tilde{u}]$.

Lemma 4.5.9. *There is a complex structure j on T and a holomorphic map $\tilde{u} : T \rightarrow \tilde{M}$ such that $\tilde{u}_*[T] = C$.*

Proof. Lemma 4.5.8 allows us to apply Lemma 4.5.7, which yields map $\tilde{u} : T \rightarrow \tilde{M}$. More precisely, the complex structure on T is the one obtained from uniformization

of the region bounded by the simple lift of $\gamma = \partial[u]$ at the end of the construction in 4.5.7, as a Riemann surface with boundary and punctures (at the nonsmooth points of the loop). \square

Pushing the map \tilde{u} from Lemma 4.5.9 down to M , we obtain the existence of a holomorphic representative in the homotopy class ϕ , for a particular complex structure on the domain.

4.5.3 The moduli space of holomorphic representatives with varying conformal structure

Let $\mathcal{M}(\phi, j)$ denote the moduli space of (j, J_M) -holomorphic maps $u : T \rightarrow M$ in the homotopy class ϕ . Let $\mathcal{M}(\phi) = \bigcup_j \mathcal{M}(\phi, j)$ denote the moduli space of such maps with varying conformal structure on the domain. Let \mathcal{R}^{k+3} denote the moduli space of conformal structures on the disk with $(k + 3)$ boundary punctures. There is a natural map $\pi : \mathcal{M}(\phi) \rightarrow \mathcal{R}^{k+3}$ which forgets the map.

Lemma 4.5.10. *For any $u \in \mathcal{M}(\phi, j)$, we have $\text{ind } D_{u,(j,J_M)} = 0$ and $\ker D_{u,(j,J_M)} = 0$.*

Proof. Let the intersection points $q_{a,i}, x_1, \dots, x_k, q_{b,j}$ be regarded as positive punctures and let $q_{a+b,h}$ be regarded as a negative puncture. Then by the conventions for Maslov index, we have that all of these intersection points have index 0, for as we go $L(0) \rightarrow L(n), L(n) \rightarrow \tau L(n), \dots, \tau^k L(n) \rightarrow L(n + m)$, and $L(0) \rightarrow L(n + m)$, the Lagrangian tangent space tilts clockwise by a small amount. Then by Proposition 11.13 of [34] (with $|\Sigma^-| = 1$ for the negative puncture), we have $\text{ind } D_{u,(j,J_M)} = 0$.

Furthermore, the operator $D_{u,(j,J_M)}$ is a Cauchy–Riemann operator acting on the line bundle u^*TM , so the results of Section (11d) of [34] apply. The hypotheses of Lemma 11.5 are satisfied with $\mu(\rho_1) = 0$ and $|\Sigma^-| = 1$, so $\ker D_{u,(j,J_M)} = 0$. \square

Lemma 4.5.11. *The map $\pi : \mathcal{M}(\phi) \rightarrow \mathcal{R}^{k+3}$ is a proper submersion of relative dimension zero (in other words, a finite covering).*

Proof. The relative dimension is the dimension of a generic fiber, which is $\text{ind } D_{u,(j,J_M)}$ for some fixed j . By the previous Lemma this is zero. Thus the dimension of $\mathcal{M}(\phi)$ is equal to that of \mathcal{R}^{k+3} , which is k .

If $u \in \mathcal{M}(\phi, j) \subset \mathcal{M}(\phi)$ is a point where the map $\pi : \mathcal{M}(\phi) \rightarrow \mathcal{R}^{k+3}$ is not a submersion, we must have $\ker D\pi \neq 0$. On the other hand $\ker D\pi$ consists of infinitesimal deformations of the map which do not change the conformal structure on the domain, and so is equal to $\ker D_{u,(j,J_M)}$, which is zero by the previous lemma. Hence π is a submersion.

The properness of π is an instance of Gromov–Floer compactness. The only thing to check is whether, as we vary $j \in \mathcal{R}^{k+3}$, any strips can break off. This is impossible because our boundary conditions do not bound any bigons in M . \square

Lemma 4.5.12. *The map $\pi : \mathcal{M}(\phi) \rightarrow \mathcal{R}^{k+3}$ has degree one.*

Proof. For this Lemma we will pass to the Gromov–Floer compactification $\bar{\pi} : \bar{\mathcal{M}}(\phi) \rightarrow \bar{\mathcal{R}}^{k+3}$. Because no bigons can break off, this compactification consists entirely of stable disks, and $\bar{\pi}$ is also a proper submersion. Hence to count the degree of π , it will suffice to count the points in the fiber of $\bar{\pi}$ over a corner of $\bar{\mathcal{R}}^{k+3}$, which is to say when the domain is a maximally degenerate stable disk.

A maximally degenerate stable disk $S = (G, (S_\alpha))$ consists of a trivalent graph $G = (V, E^{fin} \cup E^\infty)$ without cycles, with $(k+3)$ infinite edges E^∞ , and a disk S_α with three boundary punctures for each $\alpha \in V$. The boundary punctures of S_α are labeled by elements of $E^{fin} \cup E^\infty$. The elements of E^{fin} correspond to nodes of the stable disk, while the elements of E^∞ correspond to boundary punctures of the smooth domains in \mathcal{R}^{k+3} . The homotopy class ϕ determines the Lagrangian boundary conditions on each component S_α , and the asymptotic values at the boundary punctures labeled by E^∞ . The position of the nodes labeled by E^{fin} is not determined *a priori*.

Looking at the Lagrangians $L(0), L(n), \dots, L(n+m)$ shows that any three of them bound triangles, and that such triangles are determined by two of the corners. Hence by tree-induction starting at the leaves of stable disk (those S_α for which two punctures are labeled by E^∞), the positions of all the nodes are determined by ϕ , or

we run into a contradiction because no triangles consistent with the labeling exist.

Furthermore, in each homotopy class of triangles consistent with the labeling of S_α , there is at exactly one holomorphic representative.

Hence there is at most one stable map from the stable domain $S = (G, (S_\alpha))$ to M consistent with the homotopy class ϕ .

Thus we have shown that the degree of $\pi : \mathcal{M}(\phi) \rightarrow \mathcal{R}^{k+3}$ is either zero or one. On the other hand, Lemma 4.5.9 shows that $\mathcal{M}(\phi)$ is not empty, so the degree must be one. \square

Proposition 4.5.3 follows immediately from Lemmas 4.5.11 and 4.5.12.

4.6 Signs

In order to determine the signs appearing in the counts of triangles, we need to specify the brane structures on the Lagrangians $L(d)$. Since $L(d)$ fibers over a curve in the base, and its intersection with each fiber is a curve, the tangent bundle of $L(d)$ is trivial, and we can define a framing of $TL(d)$ using the vertical and horizontal tangent vectors at each point. Using this framing, we can give $L(d)$ a trivial $\text{Spin}(2)$ structure, which is induced by product of the trivial $\text{Spin}(1)$ structures on the horizontal and vertical tangent bundles.

Although we have not said much about it up until now, strictly speaking the generators of $CF^*(L(d_1), L(d_2))$ are not canonically identified with intersection points $q \in L(d_1) \cap L(d_2)$. Rather, each intersection point q gives rise to a 1-dimensional \mathbb{R} -vector space, the orientation line $o(q)$, generated by the two orientations of the point q subject to the condition that their sum is zero. In order to give a pseudo-holomorphic curve a definite sign, we must first choose trivializations of the orientation lines $o(q)$.

In the case $d_1 \leq d_2$, where all the intersections have degree 0, there is a preferred choice of trivialization for $o(q)$. Let $q \in L(d_1) \cap L(d_2)$. Then we have horizontal-vertical splittings

$$T_q L(d_i) = (T_q L(d_i))^h \oplus (T_q L(d_i))^v \tag{4.30}$$

The intersection point q has degree 0 as a morphism from $L(d_1)$ to $L(d_2)$, and moreover both $(T_q L(d_i))^h$ and $(T_q L(d_i))^v$ tilt clockwise by a small amount as we pass from $L(d_1)$ to $L(d_2)$.

Let H denote the half-plane with a negative puncture. We can define an orientation operator $D_q = D_q^h \oplus D_q^v$, acting on the product bundle $\mathbb{C} \times \mathbb{C} \rightarrow H$, where the boundary condition in the first factor is the short path $(T_q L(d_1))^h \rightarrow (T_q L(d_2))^h$, while that in the second factor is the short path $(T_q L(d_1))^v \rightarrow (T_q L(d_2))^v$. By [34], equation (11.39), we have a canonical isomorphism

$$\det(D_q) \cong o(q) \tag{4.31}$$

On the other hand, D_q is the direct sum of the operators D_q^h and D_q^v , which have vanishing kernel and cokernel. Hence

$$\det(D_q) \cong \det(D_q^h) \otimes \det(D_q^v) \cong \mathbb{R} \otimes \mathbb{R} \cong \mathbb{R} \tag{4.32}$$

where all isomorphisms are canonical. This gives us a preferred choice of isomorphism $o(q) \cong \mathbb{R}$.

Proposition 4.6.1. *Taking the preferred isomorphisms $o(q) \cong \det(D_q) \cong \mathbb{R}$ for all generators $q \in CF^0(L(d_1), L(d_2))$ for $d_1 \leq d_2$, all of the holomorphic triangles found above have positive sign.*

Proof. Let $u : S \rightarrow X(B)$ be a triangle with positive punctures at $q_1 \in CF^0(L(d_1), L(d_2))$ and $q_2 \in CF^0(L(d_2), L(d_3))$ and negative puncture at $q_0 \in CF^0(L(d_1), L(d_3))$. Let D_u denote the linearized operator at u . Gluing onto D_u the chosen orientation operators D_{q_2} and D_{q_1} in that order gives another orientation operator D'_{q_0} for the point q_0 . Since all Spin structures involved are trivial, they introduce no complication in this gluing. Since D_u , D_{q_2} , and D_{q_1} are index zero operators with vanishing kernel and cokernel, D'_{q_0} is as well, and has $\det(D'_{q_0}) \cong \mathbb{R}$ canonically. Hence the isomorphism $\det(D'_{q_0}) \cong o(q_0)$ induces the same orientation as $\det(D_{q_0}) \cong o(q_0)$.

□

Chapter 5

A tropical count of triangles

Abouzaid, Gross and Siebert have proposed a definition of a category defined from the tropical geometry of an integral affine manifold, which is meant to describe some part of the Fukaya category of the corresponding symplectic manifold. The starting point for this definition is an integral affine manifold B . The objects are then the non-negative integers, with $\text{hom}^0(d_1, d_2) = \text{span } B(\frac{1}{d_2-d_1}\mathbb{Z})$ when $d_1 < d_2$, and chains on B when $d_1 = d_2$. The composition is defined by counting a certain type of tropical curve that is balanced after addition of “tropical disks” that end on the singular locus of the affine structure.

The motivation is that the non-negative integers correspond to certain Lagrangian sections $L(n)$ of a special Lagrangian torus fibration over B , such that the intersection points between $L(d_1)$ and $L(d_2)$ lie precisely over the points in $B(\frac{1}{d_2-d_1}\mathbb{Z})$. The tropical curves then correspond to the pseudoholomorphic polygons counted in the A_∞ operations.

The Lagrangians considered above are essentially an example of this symplectic setup, so it is encouraging that our computation agrees with the expectation of Abouzaid-Gross-Siebert: The tropical triangles counted in their definition correspond closely to the pseudoholomorphic triangles found in Chapter 4

5.1 Tropical polygons

Let ∇ denote the canonical torsion-free flat connection on B associated to the affine structure. The following definition is due to Abouzaid [3]:

Definition 4. Let q_0, q_1, \dots, q_k be points of B , with $q_j \in B(\frac{1}{d_j}\mathbb{Z})$. Let Γ be a metric ribbon tree with $k+1$ infinite edges. One infinite edge, the *root*, is labeled with x_0 and it is the output. The other k infinite edges, the *leaves*, are labeled with q_1, \dots, q_k in counterclockwise order and these are the inputs. Assign to the region between q_j and q_{j+1} the weight $\sum_{i=1}^j d_i$, and give the region between q_0 and q_1 weight 0. Orient the tree upward from the root, so that each edge has a “left” and a “right” side coming from the ribbon structure. To each edge e , assign a weight w'_e given by the weight on the left side of e minus the weight on the right side of e . Define a corrected weight w_e by

$$\begin{aligned} w_e &= 0 \text{ if } w'_e < 0 \text{ and } e \text{ contains a leaf} \\ w_e &= 0 \text{ if } w'_e > 0 \text{ and } e \text{ contains the root} \\ w_e &= w'_e \text{ otherwise} \end{aligned} \tag{5.1}$$

Then a *tropical polygon* modeled on Γ is a map $u : \Gamma \rightarrow B$ such that:

1. u maps the root to q_0 and the j -th leaf to q_j ;
2. on the edge e , the tangent vector \dot{u}_e to the component u_e satisfies

$$\nabla_{\dot{u}_e} \dot{u}_e = w_e \dot{u}_e \tag{5.2}$$

with $\dot{u}_e = 0$ at the root and leaves; this differential equation only holds outside a finite set of points where tropical disks are attached;

3. there exists a finite collection of tropical disks v_1, \dots, v_N such that the union $u \cup \{v_1, \dots, v_N\}$ is balanced; the balancing condition at a vertex x is the vanishing of the sum of the derivative vectors \dot{u}_e of the various components of u incident at x and the integral tangent vectors to v_i at x , oriented toward the vertex.

To unpack this definition, let us restate it in the simplest case, which is that of tropical triangles (this mainly simplifies the issues regarding weights):

Proposition 5.1.1. *Let $q_1 \in B(\frac{1}{n}\mathbb{Z})$ and $q_2 \in B(\frac{1}{m}\mathbb{Z})$, with $n > 0$ and $m > 0$. Let $q_0 \in B(\frac{1}{n+m}\mathbb{Z})$. Let Γ be the ribbon tree with one vertex and three infinite edges. Then a tropical triangle modeled on Γ consists of three maps $u_0 : [0, \infty) \rightarrow B$, $u_1 : (-\infty, 0] \rightarrow B$, $u_2 : (-\infty, 0] \rightarrow B$ such that*

1. $u_0 \equiv q_0$ is a constant map;
2. $u_1(0) = u_2(0) = q_0$;
3. $u_1(-\infty) = q_1$, $u_2(-\infty) = q_2$;
4. We have $\nabla_{\dot{u}_1} \dot{u}_1 = n\dot{u}_1$ and $\nabla_{\dot{u}_2} \dot{u}_2 = m\dot{u}_2$, outside of a finite set of points where tropical disks are attached;
5. there exists a finite collection of tropical disks v_1, \dots, v_N such that the balancing condition holds.

We can also unpack the equation $\nabla_{\dot{u}} \dot{u} = n\dot{u}$. The key outcome of the following Lemmas is the insight that the tangent vector \dot{u} increases by n times the distance the path u travels.

Lemma 5.1.2. *Let $\gamma : [0, 1] \rightarrow B$ be a geodesic for ∇ , which means $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$. Define $u : (-\infty, 0] \rightarrow B$ by $u(t) = \gamma(\exp(nt))$. Then at the point $x = \gamma(s)$, we have $\dot{u} = ns\dot{\gamma}$, and $\nabla_{\dot{u}} \dot{u} = n\dot{u}$.*

Proof. We set $s = \exp(nt)$. The equation $\dot{u} = ns\dot{\gamma}$ is just the chain rule (note that $\dot{u} = du/dt$ while $\dot{\gamma} = d\gamma/ds$). We have

$$\nabla_{\dot{\gamma}} \dot{u} = \nabla_{\dot{\gamma}}(ns\dot{\gamma}) = n\dot{\gamma} + ns\nabla_{\dot{\gamma}} \dot{\gamma} = n\dot{\gamma} \tag{5.3}$$

since γ is a geodesic. Multiplying this equation by ns , and using the fact that ∇ is tensorial in its subscript, we obtain $\nabla_{\dot{u}} \dot{u} = n\dot{u}$. \square

Lemma 5.1.3. *Let $u : [a, b] \rightarrow B$ solve $\nabla_{\dot{u}}\dot{u} = n\dot{u}$. Then if we patch together affine charts along u to embed a neighborhood of u into \mathbb{R}^n , we have*

$$\dot{u}(b) - \dot{u}(a) = n(u(b) - u(a)) \quad (5.4)$$

where we use the affine structure of \mathbb{R}^n to take differences of points and vectors at different points.

Proof. Define $\gamma : [\exp(na), \exp(nb)] \rightarrow B$ by $\gamma(s) = u((\log s)/n)$, so that γ is a geodesic. We can embed a neighborhood of γ into \mathbb{R}^n by patching together affine coordinate charts along γ . Using addition in this embedding, we can write $\gamma(s) = \gamma(\exp(na)) + (s - \exp(na))\dot{\gamma}$. We have $\dot{u} = n s \dot{\gamma}$, and

$$\dot{u}(b) - \dot{u}(a) = n(\exp(nb) - \exp(na))\dot{\gamma} = n(\gamma(\exp(nb)) - \gamma(\exp(na))) = n(u(b) - u(a)) \quad (5.5)$$

□

One more thing to note regarding these tangent vectors \dot{u} is how they represent homology classes on the torus fibers of the fibration over B .

When considering a tropical curve C^{trop} corresponding to a closed holomorphic curve C , each edge of the tropical curve carries an integral tangent vector, which morally represents the class $[C \cap T_b^2] \in H_1(T_b^2; \mathbb{Z})$ that measures how the holomorphic curve intersects the torus fiber $T_b^2 = \pi^{-1}(b)$. Because the curve is closed, this class is locally constant along each edge of C^{trop} .

When considering a tropical curve C^{trop} representing a holomorphic curve C with boundary on Lagrangian sections $L(i), L(j)$, the intersection of C with T_b^2 would morally be a path on T_b^2 from $L(i)_b$ to $L(j)_b$, where $L(i)_b$ is intersection of T_b^2 and $L(i)$. Let $A(L(i)_b, L(j)_b) \subset H_1(T_b^2, \{L(i)_b, L(j)_b\}; \mathbb{Z})$ be the subset consisting of such cycles, which could also be described as the preimage of $L(j)_b - L(i)_b \in H_0(\{L(i)_b, L(j)_b\}; \mathbb{Z})$ under the boundary homomorphism. Hence $A(L(i)_b, L(j)_b)$ is a torsor for the kernel of that homomorphism, which is $H_1(T_b^2; \mathbb{Z})$. Using the group structure on T_b^2 , we can

identify $A(L(i)_b, L(j)_b)$ with the coset

$$[L(j)_b - L(i)_b] + H_1(T_b^2; \mathbb{Z}) \subset H_1(T_b^2; \mathbb{R}) \quad (5.6)$$

On the other hand, there is an isomorphism $(T_b B)_\mathbb{R} \cong H_1(T_b^2; \mathbb{R})$. Hence the class of $[C \cap T_b^2]$ can be regarded as a tangent vector to the base, which is in general real and varies along the tropical curve as $L(i)$ and $L(j)$ move relative to one another. The tangent vector \dot{u} to the tropical curve is this class.

The balancing condition at a vertex b of the tropical polygon amounts to requiring that the three paths $L(i)_b \rightarrow L(j)_b$, $L(j)_b \rightarrow L(k)_b$ and $L(k)_b \rightarrow L(i)_b$ form a contractible loop. At a point where a tropical disk is attached, the path $L(i)_b \rightarrow L(j)_b$ changes discontinuously by a loop in the homology class in $H_1(T_b^2; \mathbb{Z})$ corresponding to the integer tangent vector to the tropical disk.

The last thing to describe for tropical polygons is their multiplicities. For any tropical disk v , the Gross–Siebert theory can associate a multiplicity $m(v)$, which is a virtual count of holomorphic disks corresponding to v .

There is also a multiplicity coming from the different ways to attach a disk v to the tropical polygon. If one incoming edge of the polygon has tangent vector \dot{u}_e corresponding to a path $\gamma_1 : L(i)_b \rightarrow L(j)_b$ on T_b^2 , and the disk has tangent vector w corresponding to a loop γ_2 on T_b^2 , there are $|\gamma_1 \cdot \gamma_2| = |\det(\dot{u}_e, w)|$ points where the disk can be attached, assuming this determinant is an integer (as it is in the special case below). Otherwise, one must look carefully at exactly where on the torus the paths γ_1 and γ_2 are located.

In general, the multiplicity of a tropical polygon will have both the Gross–Siebert factors $m(v)$ counting how many holomorphic disks are in each class, as well as simpler factors counting how many ways these classes of disks can be attached.

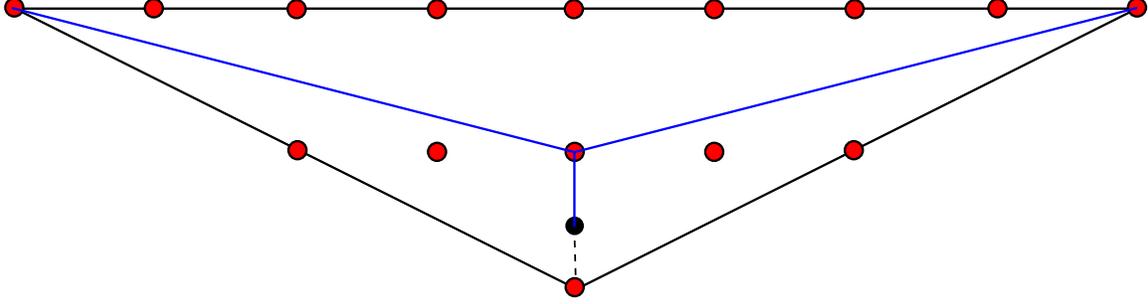


Figure 5-1: A tropical triangle.

5.2 Tropical triangles for (\mathbb{CP}^2, D)

We now write out explicitly the tropical curves contributing to the triangle products in the case of (\mathbb{CP}^2, D) . Let us use coordinates (η, ξ) where the point $q_{a,i} \in B(\frac{1}{n}\mathbb{Z})$ has coordinates $(a/n, -i/n)$

There is one family of simple tropical disks that emanate from the singularity on the $\eta = 0$ line in the vertical direction. Their primitive tangent vectors are $\pm(0, 1)$.

Figure 5-1 shows the the tropical triangle representing the contribution of y^2p to the product of x^2 and z^2 . This triangle has multiplicity 2. The singularity of the affine structure is placed so as to emphasize the tropical disk ending at the singularity.

Proposition 5.2.1. *Let $n > 0$ and $m > 0$, and take $q_{a,i} \in B(\frac{1}{n}\mathbb{Z})$, $q_{b,j} \in B(\frac{1}{m}\mathbb{Z})$, and $q_{a+b,h} \in B(\frac{1}{n+m}\mathbb{Z})$.*

Suppose a and b have different signs, and let $k = \min(|a|, |b|)$, and $h = i + j + s$. If $0 \leq s \leq k$, there is one tropical triangle connecting these three points. It is balanced after the addition of either s or $k - s$ tropical disks, depending on the position of the singularity. The multiplicity of this curve is $\binom{k}{s}$. If s does not lie in this range, there is no triangle.

Suppose that a and b have the same sign. Then unless $h = i + j$ there is no triangle, and when $h = i + j$ there is exactly one, which is represented geometrically by the line segment joining $q_{a,i}$ and $q_{b,j}$.

Proof. When a and b have the same sign the tropical triangle can have no tropical disks attached to it. Therefore the tropical triangle is simply a line segment which

passes through the three points, and this only exists if $q_{a+b,h}$ lies on the line between $q_{a,i}$ and $q_{b,j}$.

As for when a and b have different signs, let us consider the case $a \leq 0$, $b \geq 0$, $a+b \geq 0$, and hence $k = -a$. The other cases are related to this by obvious reflections and renamings.

By Proposition 5.1.1, a tropical triangle consists essentially of two maps $u_1, u_2 : (-\infty, 0] \rightarrow B$, together with some copies of the tropical disk and their multiples.

- The leg u_2 of the tree connecting $q_{b,j}$ to $q_{a+b,h}$ cannot have any tropical disks attached, since both endpoints lie on the same side of the line $\eta = 0$. We can apply Lemma 5.1.3 to obtain the tangent vector \dot{u}_2 at $q_{a+b,h}$ as m times the difference between the endpoints, or

$$\dot{u}_2(q_{a+b,h}) = m(q_{a+b,h} - q_{b,j}) = \left(\frac{ma - nb}{n + m}, \frac{-(mi - nj + ms)}{n + m} \right) \quad (5.7)$$

- The leg u_1 of the tree connecting $q_{a,i}$ to $q_{a+b,h}$ crosses the line $\eta = 0$ at some point x , where tropical disks can be attached. It may bend there, and continue on to $q_{a+b,h}$, where the balancing condition $\dot{u}_1 + \dot{u}_2 = 0$ must hold. This shows that the portion of u_1 connecting x to $q_{a+b,h}$ must be parallel to u_2 . Hence x must be on the line joining $q_{b,j}$ and $q_{a+b,h}$. We obtain the position of x :

$$x : (\eta, \xi) = (0, (-aj + bi + bs)/(ma - nb)) \quad (5.8)$$

- By Lemma 5.1.3 at x the tangent vector \dot{u}_1 is given by n times the difference of the endpoints x and $q_{a,i}$:

$$\dot{u}_1(x)_L = n(x - q_{a,i}) = (-a, [a(mi - nj) + nbs]/(ma - nb)) \quad (5.9)$$

we use the subscript L to denote this is the tangent vector coming from the left.

- If the singularity of the affine structure occurs below the point x , then tropical disks propagate upward in the direction $(0, 1)$. Adding the vector $(0, s)$ to

$\dot{u}_1(x)_L$, we obtain $\dot{u}_2(x)_R$, the tangent vector from the right,

$$\dot{u}_1(x)_R = \dot{u}_1(x)_L + (0, s) = (-a, a(mi - nj + ms)/(ma - nb)) \quad (5.10)$$

which is parallel to $\dot{u}_2(q_{a+b,h})$, as it must be. This shows that we must attach a collection of tropical disks whose total weight is s .

- If the singularity of the affine structure occurs above x , then tropical disks propagate in the direction $(0, -1)$, but there is also the monodromy to be taken into account. First we must act on $\dot{u}_1(x)_L$ by the monodromy $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ to get

$$M\dot{u}_1(x)_L = (-a, [a(mi - nj) + nbs]/(ma - nb) - a) \quad (5.11)$$

Adding the vector $(0, -(k - s))$ to this, with $k = -a$, gives the same result as before for $\dot{u}_1(x)_R$. However, in this case we are attaching a collection of disks with total weight $(k - s)$.

- The leg u_1 propagates in the direction $\dot{u}_1(x)_R$ from x to $q_{a+b,h}$. As it does so, the tangent vector \dot{u}_1 increases by $\Delta = n(q_{a+b,h} - x)$, which is parallel to $\dot{u}_1(x)_R$. By comparing affine lengths, we have the proportion

$$\dot{u}_1(x)_R : \Delta = [-a/n] : [(a + b)/(n + m)] \quad (5.12)$$

and

$$(\dot{u}_1(x)_R + \Delta) : \dot{u}_1(x)_R = [(a + b)/(n + m) - a/n] : [-a/n] = [-(ma - nb)/(n + m)] : [-a] \quad (5.13)$$

Thus

$$\dot{u}_1(q_{a+b,h}) = \left(-\frac{ma - nb}{n + m}, \frac{mi - nj + ms}{n + m} \right) = -\dot{u}_2(q_{a+b,h}) \quad (5.14)$$

Verifying the balancing condition at $q_{a+b,h}$.

Now that we know which tropical curves contribute, we must compute their multiplicities. The tropical curve constructed above uses the tropical disk s or $(k - s)$ times. This means that either we attach a single simple disk s times, or we attach some multiple covers of the disk in some fashion as to achieve a total multiplicity of s . In general, the count of multiple covers of the disk is obtained tropically from the Gross–Siebert program. Since the simple disks give the desired result, we claim that multiple covers of the disk do not count.

Attaching simple disks does introduce a multiplicity, since there are multiple places to attach this disk. In fact, we have $\det(\dot{u}_1(x)_L, (0, 1)) = -a = k$, so there are a total of k places for disks to be attached. Thus we get the multiplicity $\binom{k}{s}$ or $\binom{k}{k-s}$, which are equal and give the desired result.

□

The fact that the multiplicities $\binom{k}{s}$ and $\binom{k}{k-s}$ are equal is an illustration of the general phenomenon that the exact position of the singularity along its invariant line does not matter for tropical curve counts.

Chapter 6

Parallel monodromy–invariant directions

In this section we collect some remarks about a class of affine manifolds to which the results obtained for $(\mathbb{C}\mathbb{P}^2, D)$ naturally generalize.

Definition 5. Let B be a two-dimensional affine manifold with singularities x_1, \dots, x_n . We say that B has *parallel monodromy-invariant directions* if there is a line field Ξ on $B \setminus \{x_1, \dots, x_n\}$ that is constant with respect to the affine structure: locally Ξ has a nonvanishing section X such that $\nabla X = 0$.

It follows immediately from the definition that each singularity has a monodromy invariant direction in the direction of Ξ . Furthermore, if B is integral affine, then under certain topological conditions we have a globally defined affine coordinate:

Proposition 6.0.2. *Let B be an orientable two-dimensional integral affine manifold with parallel monodromy-invariant directions, such that the line field Ξ is orientable, and with $H^1(B; \mathbb{R}) = 0$. Then there is a function $\eta : B \rightarrow \mathbb{R}$ that is integral affine linear in each coordinate chart, and such that $d\eta(\Xi) = 0$.*

Proof. Let $X_1, X_2 \in (T_b B)_{\mathbb{Z}}$ be an integral basis of the tangent space to $b \in B$, where $X_2 \in \Xi$. Since parallel transport around a singularity must preserve the subspace Ξ ,

in this basis its matrix has the form

$$A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \quad (6.1)$$

Since this matrix is invertible over \mathbb{Z} , we have $a, d \in \{\pm 1\}$. Since Ξ is orientable, we must have $d = 1$, and since B is orientable, we must have $a = 1$ as well. Let $\alpha_1, \alpha_2 \in (T_b^*B)_{\mathbb{Z}}$ be a basis of integral 1-forms dual to X_1, X_2 . Then the action of monodromy on 1-forms is given in this basis by the matrix

$$A^* = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \quad (6.2)$$

Thus, α_1 is preserved by all monodromies and defines a global 1-form α that is constant with respect to the affine structure. Since $H^1(B; \mathbb{R}) = 0$, there is a function $\eta : B \rightarrow \mathbb{R}$ such that $d\eta = \alpha$. \square

Definition 6. Let B be a two-dimensional affine manifold with boundary, corners and singularities. We say that B has *polygonal type* if B is contractible and each boundary facet is straight with respect to the affine structure.

Now we define the relevant class of affine manifolds:

Definition 7. Let \mathcal{P}' denote the class of affine manifolds B such that

1. the affine structure on B is integral,
2. B has polygonal type,
3. B has only focus–focus singularities,
4. B has parallel monodromy invariant directions,

Proposition 6.0.3. *Any $B \in \mathcal{P}'$ satisfies the hypotheses of Proposition 6.0.2, and has a globally defined integral affine coordinate $\eta : B \rightarrow \mathbb{R}$. There exists another function $\xi : B \rightarrow \mathbb{R}$ such that $(\eta, \xi) : B \rightarrow \mathbb{R}^2$ is an embedding of B as a polygonal*

region in \mathbb{R}^2 , and such that this embedding is integral affine linear outside a set of branch cuts emanating from the singularities in a monodromy-invariant direction.

Proof. Since B is contractible, it is orientable and has $H^1(B; \mathbb{R}) = 0$. Since the monodromy of the focus–focus singularity leaves the orientation of the invariant line intact, Ξ is orientable.

For each singularity of B , introduce a branch cut emanating from the singularity in a monodromy-invariant direction. Let ξ be an integral affine coordinate complementary to η in any coordinate chart. Then continuation of ξ outside the branch cuts yields a well-defined function $\xi : B \rightarrow \mathbb{R}$.

The image of B under the embedding is a region whose sides are straight with respect to the affine structure of \mathbb{R}^2 , but with apparent corners at the places where ∂B crosses a branch cut. □

In light of this proposition, we introduce one more restriction on the affine manifold

Definition 8. Let \mathcal{P} denote the class of affine manifolds $B \in \mathcal{P}'$ for which additionally

- (5) The corners of B occur at extreme values of $\eta : B \rightarrow \mathbb{R}$.

We now summarize how the various aspects of the construction and computation for $(\mathbb{C}\mathbb{P}^2, D)$ generalize to manifolds in the class \mathcal{P} .

6.1 Symplectic forms

Let x_1, \dots, x_n denote the singularities of B . Let $[a_0, a_{n+1}]$ be the image of B under η , and let $a_i = \eta(x_i)$. We split up B along the monodromy invariant lines of each singularity, and obtain intervals $I_i = [a_i + \epsilon, a_{i+1} - \epsilon]$, with fibrations $X(B_i) \rightarrow X(I_i)$. $X(B_i)$ has complex coordinates (w, z_i) corresponding to (η, ξ) from Proposition 6.0.3, while the coordinate on $X(I_i)$ is w .

Each piece B_i is an affine manifold whose horizontal boundary consists of two straight lines (since B has no corners but at the extreme values of η), and hence we are in the situation of section 3.1. We obtain a symplectic form for which the

symplectic connection of $X(B_i) \rightarrow X(I_i)$ foliates the horizontal boundary facets of $X(B_i)$.

Corresponding to each focus–focus singularity, we glue in a Lefschetz singularity. The discussion in section 3.2 applies directly. The result is a manifold $X(B)$ with a Lefschetz fibration $w : X(B) \rightarrow X(I)$, and such that the horizontal boundary faces of $X(B)$ are foliated by the symplectic connection. Let w_1, \dots, w_n denote the critical values of $w : X(B) \rightarrow X(I)$.

6.2 Lagrangian submanifolds

As before, the construction of Lagrangian sections proceeds by taking paths in the base and a Lagrangian in the fiber, and sweeping out a Lagrangian in the total space by parallel transport. Potentially, we have more freedom than in the mirror to $\mathbb{C}\mathbb{P}^2$.

The Lagrangian submanifolds we consider have boundary conditions given by a complex curve in $\partial X(B)$ along each boundary face. This gives two curves Σ_0 and Σ_1 for the bottom and top horizontal boundary faces. If B has vertical boundary faces, we also have curves corresponding to these in the vertical boundary; these are just particular fibers M_0, M_1 of the fibration $X(B) \rightarrow X(I)$. If B has a corner rather than a vertical boundary face, we still have a distinguished fiber M , and the Lagrangians are required to intersect this fiber in a chosen curve.

Thus, a Lagrangian submanifold may be constructed by taking a Lagrangian L_0 in M_0 , and a path ℓ in the base joining M_0 to M_1 , and taking the parallel transport. If M_0 corresponds to a corner, we have only one choice for L_0 , and if M_1 corresponds to a corner, this imposes a constraint on L_0 and ℓ . In order to obtain sections of the torus fibration, we choose L_0 to be a curve which is a section of the fibration of the fiber by circles of constant ξ , and ℓ to be a section of the fibration of the base by circles of constant η . Thus when drawing $X(I)$ as an annulus, ℓ appears as a spiral.

In some situations, we are free to choose all the parameters independently. For example, if B is a four-sided affine manifold with two vertical sides and two singularities, we can choose independently

1. the number of times the initial Lagrangian L_0 winds around the fiber M_0 ,
2. the number of times ℓ winds around the base between the M_0 and the first singularity,
3. the number of times ℓ winds around the base between the two singularities, and
4. the number of times ℓ winds around the base between the second singularity and M_1 .

This gives rise to a 4-parameter family of Lagrangians. Under mirror symmetry, all of them correspond to line bundles. We claim that $X(B)$ is a mirror to the degree 6 del Pezzo surface X_6 with a 4-component anticanonical divisor. The 4 parameters correspond to $\text{Pic}(X_6) \cong \mathbb{Z}^4$.

Though we can construct many Lagrangians this way, in order to compute Floer cohomology and identify the basis with $B(\frac{1}{d}\mathbb{Z})$, we must choose a family of Lagrangians $\{L(d)\}_{d \in \mathbb{Z}}$ corresponding to the tensor powers of a polarization. First we choose $\ell(0)$ as a reference path in the base, over which lies $L(0)$, a Lagrangian satisfying the boundary conditions. We take $\ell(1)$ to be a certain path in the base: the number of times that $\ell(1)$ must wind between the singularities and the vertical boundaries/corners is determined by B : it is essential that the number of turns $\ell(1)$ makes between two consecutive singularities (or between a boundary and the neighboring singularity) is the affine width of the corresponding portion of B . An equivalent condition is that the $\ell(1)$ winds at unit speed. If B has a vertical boundary face rather than a corner, the intersection of $L(1)$ with the fiber at that boundary must be a curve that, relative to $L(0)$, makes a number of turns equal to the affine length of the corresponding vertical boundary face. Then we choose $\ell(d)$ to be a path in the base whose slope is d times the slope of $\ell(1)$, relative to $\ell(0)$. We also make sure that in the fiber, the slope $L(d)$ is d times the slope of $L(1)$ (relative to $L(0)$ in each fiber).

6.3 Holomorphic and tropical triangles

Having chosen a family of Lagrangians $\{L(d)\}_{d \in \mathbb{Z}}$ corresponding to the powers of a polarization, we find again that $HF^*(L(d_1), L(d_2))$ is concentrated in degree 0 when $d_1 \leq d_2$. The techniques of Chapter 4 allow us to compute the holomorphic triangles contributing to the multiplication

$$HF^*(L(d_2), L(d_3)) \otimes HF^*(L(d_1), L(d_2)) \rightarrow HF^*(L(d_1), L(d_3)) \quad (6.3)$$

Looking at the winding numbers of the Lagrangians in the base once again yields an auxiliary \mathbb{Z} -grading. For fixed values of this \mathbb{Z} -grading on the input, one can determine the number of times that triangles contributing to the product cover the critical values w_1, \dots, w_n ; call these numbers k_1, \dots, k_n . Then the degeneration process breaks the triangle into $k = \sum_{i=1}^n k_i$ copies of the fibration over a disk with single critical value, as well a trivial fibration over a $(k+3)$ -gon. Over the disks the count of sections is 1, while the analysis of sections over the $(k+3)$ -gon still goes through because, in the fiber, the Lagrangian boundary condition is still a sequence of curves on the cylinder whose slope changes monotonically. Hence the matrix coefficients of this product are binomial coefficients of the form $\binom{k}{s}$.

In this degeneration argument, the Lefschetz singularities that come from different focus-focus singularities are not distinguished, while in the case of tropical triangles, different singularities of the affine structure contribute differently to the tropical curves. We find that this family of triangles, with total count $\binom{k}{s}$, corresponds to several tropical triangles T_{s_1, \dots, s_n} with $s_1 + \dots + s_n = s$, indexed by ordered partitions of s , with 0 allowed as a part (of which there are $\binom{s+n}{s}$). The triangle T_{s_1, \dots, s_n} uses the tropical disk emanating from the i -th singularity either s_i or $k_i - s_i$ times, and the multiplicity of T_{s_1, \dots, s_n} is $\binom{k_1}{s_1} \binom{k_2}{s_2} \dots \binom{k_n}{s_n}$. The equality of the total counts

$$\binom{k}{s} = \sum_{\{s_1, \dots, s_n \mid s_i \geq 0, \sum_{i=1}^n s_i = s\}} \prod_{i=1}^n \binom{k_i}{s_i} \quad (6.4)$$

follows from comparing the coefficients of x^s in the equation

$$(1+x)^k = \prod_{i=1}^n (1+x)^{k_i} \tag{6.5}$$

Chapter 7

Mirrors to divisor complements

In this section we examine the relationship between the wrapped Floer cohomology of our Lagrangians $L(d)$ and the cohomology of coherent sheaves on complements of components of the anticanonical divisor in $\mathbb{C}\mathbb{P}^2$. Let $D = C \cup L$ denote the anticanonical divisor which is the union of a conic C and a line L . Then we can consider the divisor complements $U_D = \mathbb{C}\mathbb{P}^2 \setminus D$, $U_C = \mathbb{C}\mathbb{P}^2 \setminus C$, and $U_L = \mathbb{C}\mathbb{P}^2 \setminus L$. Then the torus fibration $\mathbb{C}\mathbb{P}^2 \setminus D$ can be restricted to such a complement, and T-duality gives the same space X^\vee as before, but with a different superpotential, reflecting the counts of holomorphic disks intersecting the remaining components of the anticanonical divisor.

Once again, the cohomology of coherent sheaves $\mathcal{O}(d)$ corresponds to Floer cohomology of the Lagrangian submanifolds $L(d)$.

Removing a divisor D from a compact variety X changes the cohomology of a coherent sheaf \mathcal{F} , since, for example, sections of \mathcal{F} with poles along D are regular on the complement $U = X \setminus D$, so that $H^0(U, \mathcal{F})$ is not finitely generated in general.

Variety	Anticanonical divisor	Mirror space	Superpotential
$\mathbb{C}\mathbb{P}^2$	$D = C \cup L$	$X^\vee = \{(u, v) \mid uv \neq 1\}$	$W = u + \frac{e^{-\Lambda} v^2}{uv-1}$
$U_L = \mathbb{C}\mathbb{P}^2 \setminus L$	$C \setminus (C \cap L)$	X^\vee	$W_L = u$
$U_C = \mathbb{C}\mathbb{P}^2 \setminus C$	$L \setminus (L \cap C)$	X^\vee	$W_C = \frac{e^{-\Lambda} v^2}{uv-1}$
$U_D = \mathbb{C}\mathbb{P}^2 \setminus D$	\emptyset	X^\vee	$W_D = 0$

Table 7.1: Mirrors to divisor complements.

On the symplectic side, changing the superpotential by dropping a term modifies the boundary condition for our Lagrangian submanifolds $L(d)$. Some parts of $L(d)$ that were required to lie on the fiber of W are no longer so constrained, and it is appropriate to *wrap* these parts of $L(d)$. The algebraic structure associated to $L(d)$ is then *wrapped Floer cohomology* $HW^*(L(d_1), L(d_2))$, which is the limit $\lim_{w \rightarrow \infty} HF^*(\phi_{wH}(L(d_1)), L(d_2))$, where H is an appropriate Hamiltonian function (a more precise definition is given below). The limit $HW^*(L(d_1), L(d_2))$ will not be finitely generated in general, since it potentially contains trajectories of H joining $L(d_1)$ to $L(d_2)$ of *any* length. The general theory of wrapped Floer cohomology is developed in [4].

7.1 Algebraic motivation

In order to motivate the symplectic constructions of wrapped Floer cohomology, it is useful to understand the algebraic side first. The starting point is the following proposition ([33], statement (1.10)).

Proposition 7.1.1. *Let X be a smooth quasiprojective variety over \mathbb{C} , $Y \subset X$ a hypersurface, and $U = X \setminus Y$ the complement. Write $Y = s^{-1}(0)$, where s is the canonical section of the line bundle $\mathcal{L} = \mathcal{O}_X(Y)$. Let \mathcal{F} be a coherent sheaf on X . Multiplication by s defines an inductive system*

$$H^*(X, \mathcal{F} \otimes \mathcal{L}^{r-1}) \longrightarrow H^*(X, \mathcal{F} \otimes \mathcal{L}^r) \longrightarrow H^*(X, \mathcal{F} \otimes \mathcal{L}^{r+1}) \longrightarrow \dots \quad (7.1)$$

and the limit is

$$\lim_{r \rightarrow \infty} H^*(X, \mathcal{F} \otimes \mathcal{L}^r) \cong H^*(U, \mathcal{F}|_U) \quad (7.2)$$

We spell out the application of this proposition to each of the cases we consider

- U_L : Since $L = \{y = 0\}$ is a line, we identify $\mathcal{L} \cong \mathcal{O}(1)$ and take $s = y$. Thus

$$H^*(U_L, \mathcal{O}(d)) \cong \lim_{r \rightarrow \infty} H^*(\mathbb{CP}^2, \mathcal{O}(d+r)) \quad (7.3)$$

where the limit is formed with respect to multiplication by y . An element of $H^0(U_L, \mathcal{O}(d))$ is a rational function $f(x, y, z)/y^r$, where f is a homogeneous polynomial of degree $d + r$.

- U_C : Since $C = \{xz - y^2 = 0\}$ is a conic, we identify $\mathcal{L} \cong \mathcal{O}(2)$ and take $s = p = xz - y^2$. Thus

$$H^*(U_C, \mathcal{O}(d)) \cong \lim_{r \rightarrow \infty} H^*(\mathbb{CP}^2, \mathcal{O}(d + 2r)) \quad (7.4)$$

where the limit is formed with respect to multiplication by $p = xz - y^2$. An element of $H^0(U_C, \mathcal{O}(d))$ is a rational function $f(x, y, z)/p^r$, where f is a homogeneous polynomial of degree $d + 2r$.

- U_D : Since $D = \{yp = xyz - y^3 = 0\}$ is a cubic, we identify $\mathcal{L} \cong \mathcal{O}(3)$ and take $s = yp$. Thus

$$H^*(U_D, \mathcal{O}(d)) \cong \lim_{r \rightarrow \infty} H^*(\mathbb{CP}^2, \mathcal{O}(d + 3r)) \quad (7.5)$$

where the limit is formed with respect to multiplication by yp . An element of $H^0(U_D, \mathcal{O}(d))$ is a rational function $f(x, y, z)/(yp)^r$, where f is a homogeneous polynomial of degree $d + 3r$.

For the purposes of computation, a useful simplification comes from noting that the line bundles $\mathcal{O}(d)$ may become isomorphic over the complements.

- U_L : Since $U_L \cong \mathbb{C}^2$, all the line bundles $\mathcal{O}(d)$ are isomorphic over it.
- U_C : The complement of a smooth conic in \mathbb{CP}^2 has $H^2(U_C; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, generated by $c_1(\mathcal{O}(1))$. The defining section $p : \mathcal{O} \rightarrow \mathcal{O}(2)$ is an isomorphism over U_C , and so $\text{Pic}(U_C) \cong \mathbb{Z}/2\mathbb{Z}$ as well.
- U_D : Since $U_D \subset U_L$, all the line bundles $\mathcal{O}(d)$ are isomorphic over it as well.

7.2 Wrapping

In this section we describe the geometric setup for wrapped Floer cohomology in the mirrors of U_L, U_C , and U_D .

7.2.1 Completions

Wrapped Floer cohomology is formulated in terms of noncompact manifolds containing noncompact Lagrangian submanifolds. These can be defined as completions of compact manifolds with boundary.

The starting point for all three cases is the original Lefschetz fibration $X(B) \rightarrow X(I)$ containing the Lagrangians $\{L(d)\}_{d \in \mathbb{Z}}$. The symplectic form constructed in section 3 has the defect that it blows up at the corners of B . We could potentially work with it directly, but the technically safest way to deal with it is to simply cut these corners off by restricting the fibration to a sub-annulus of $X(I)$. We remark that it is clear that the results of the previous sections all carry over to this manifold without change: nowhere was the behavior at the corners essentially used other than in motivating the restriction on how the Lagrangians $L(d)$ should behave near the corners.

Remark 8. If one wanted a mirror interpretation of cutting off the corners of B and completing, it would be blowing up the intersections $C \cap L$, and then removing the total inverse image of either L , C , or D , which is the same as just removing L , C , or D .

We define completions \hat{X}_L , \hat{X}_C , and \hat{X}_D .

- L : The manifold \hat{X}_L retains a boundary component at the top horizontal boundary, corresponding to the fiber of the superpotential $W_L = u$. The fibers of $X(B) \rightarrow X(I)$ are completed at the other, bottom, end. The base annulus $X(I)$ is completed to a cylinder $\hat{X}(I)$, and the fibration is extended over this cylinder.

- C : The manifold \hat{X}_C retains a boundary component at the bottom horizontal boundary, corresponding to the fiber of the superpotential $W_C = \frac{e^{-\Lambda} v^2}{uv-1}$. The fibers are completed at the other, top, end. The base annulus is completed to a cylinder $\hat{X}(I)$, and the fibration is extended over this cylinder.
- D : The manifold \hat{X}_D has no boundary, and the fibers are completed at both ends. The base annulus is completed to a cylinder $\hat{X}(I)$, and the fibration is extended over this cylinder.

The torus fibration on $X(B)$ extends to these completions, and yields torus fibrations over completed bases $\hat{B}_L, \hat{B}_C, \hat{B}_D$.

- \hat{B}_L is a half-plane with singular affine structure given by removing the bottom boundary from B and extending in that direction.
- \hat{B}_C is a half-plane with singular affine structure given by removing the top boundary from B and extending in that direction.
- \hat{B}_D is an entire plane with singular affine structure given by removing all boundaries from B and extending in all directions.

The Lagrangian submanifolds $L(d)$ are extended to $\hat{L}(d)$; In all cases, we extend $L(d)$ into whatever ends are attached so as to be invariant under the Liouville flow within the end. However, in the case of L , respectively C , we still have the boundary condition that $\hat{L}(d)$ is required to end on Σ_1 (the complex hypersurface contained in the top boundary), respectively Σ_0 (contained in the bottom boundary).

7.2.2 Hamiltonians

The most crucial difference between the three cases comes from the choices of Hamiltonians that are to be used to perform the wrapping. The Hamiltonians we consider are the sum of contributions from the base and the fiber.

Let $H_b : \hat{X}(B) \rightarrow \mathbb{R}$ be the pullback of a function on the base cylinder which is a function of the radial coordinate $\eta = \log |w|$ only. Writing the symplectic form on

the base as $d\rho \wedge d\theta$, where ρ is a function of η , we take H_b to be a convex function of ρ on the compact part $X(I)$, and linear in ρ on the ends. We also require $H_b \geq 0$, with minimum on the central circle $\eta = 0$. Since dH_b vanishes on the fibers, X_{H_b} is horizontal.

The fiber Hamiltonian H_f is chosen differently in each case. The main constraint is that its differential must vanish at any boundary component which may still be present. The construction is most convenient if we assume the completion preserves the S^1 -symmetry that rotates the fibers. If μ denotes the moment map for this action, we can take H_f to be a function of μ . Since X_{H_f} is tangent to the fibers, we have

$$\{H_b, H_f\} = \omega(X_{H_b}, X_{H_f}) = 0 \tag{7.6}$$

which allows us to compute the flow of $H_b + H_f$ term by term.

We use the same base Hamiltonian H_b for all cases. The specific choice of H_f in each case is as follows.

- *L*: Let $H_{f,L} \geq 0$ be a function with a minimum at the top of the fiber, convex in μ on the compact part, and linear in μ on the bottom end.
- *C*: Let $H_{f,C} \geq 0$ be a function with a minimum at the bottom of the fiber, convex in μ on the compact part, and linear in μ on the top end.
- *D*: Let $H_{f,D} \geq 0$ be a function with a minimum in the middle of the compact part of the fiber, convex in μ on the compact part, and linear in μ on the ends.

Remark 9. The Hamiltonians we obtain as $H_b + H_f$ are not admissible in the usual sense, because they vanish at some boundaries, and, even in the case *D*, are not linear with respect to a cylindrical end. Closer to our situations are the *Lefschetz admissible* Hamiltonians considered by Mark McLean [27], that are precisely those functions on the total space of a Lefschetz fibration that are the sum of admissible Hamiltonians on the base and fiber separately.

7.2.3 Generators

Given Lagrangian submanifolds L_1, L_2 of X , equipped with Hamiltonian H , and $r \in \mathbb{R}$, we get Floer cohomology complexes $CF^*(L_1, L_2; rH)$ generated by time-1 trajectories of X_{rH} starting on L_1 and ending on L_2 . As usual the differential counts inhomogeneous pseudo-holomorphic strips. We also have continuation maps

$$CF^*(L_1, L_2; rH) \rightarrow CF^*(L_1, L_2; r'H), \quad r < r' \quad (7.7)$$

given by counting strips where the inhomogeneous term interpolates between $r'X_H$ and rX_H . At the homology level, the continuation maps form an inductive system, and we define the *wrapped Floer cohomology*

$$HW^*(L_1, L_2) = \lim_{r \rightarrow \infty} HF^*(L_1, L_2; rH) \quad (7.8)$$

Our purpose in this section is simply to set up an enumeration of the generators of $CF^*(L(d_1), L(d_2); rH)$ in each of the three cases. These generators can also be regarded as intersection points $\phi_{rH}(L(d_1)) \cap L(d_2)$. In order to make the situation as convenient as possible for our later arguments, we refine our choice of Hamiltonians so as to ensure that $\phi_{rH}(L(d))$ is actually $L(d')$ for some d' ; thus we can identify wrapped Floer cohomology generators with intersection points of our original Lagrangians. This is done by adjusting the slopes of our Hamiltonians on the ends.

- We take the base Hamiltonian H_b so that the time-1 flow completes 1 turn on the cylindrical ends of the base.
- For cases L and C , we take the fiber Hamiltonian H_f so that the time-1 flow completes 1/2 turn on the cylindrical end of the fiber.
- For case D , we take the fiber Hamiltonian H_f so that the time-1 flow completes 1/3 turn at the top of the fiber, and 1/6 turn at the bottom of the fiber.

In all cases the total Hamiltonian we use is $H = H_b + H_f$.

The way to understand the flow of H is to first apply H_f , then H_b . The flow of H_f wraps $L(d)$ in the fiber, while the flow of H_b , when it completes a loop in the base, performs the monodromy of the Lefschetz fibration around that loop, which undoes some of the wrapping due to H_f .

We can relate $\phi_H(L(d))$ to $L(d')$ as follows:

- L : we have $\phi_{rH}(L(d)) = L(d - r)$.
- C : we have $\phi_{2rH}(L(d)) = L(d - 2r)$. Note that the same *cannot* be said with r in place of $2r$; in that case the two Lagrangians intersect the bottom boundary (where no wrapping occurs) in different points.
- D : we have $\phi_{3rH}(L(d)) = L(d - 3r)$. Again the same cannot be said with r in place of $3r$.

In order to identify generators with intersection points, we perturb the boundary intersection points in a positive sense just as before. In the cases with boundary, where the Hamiltonian is supposed to have a minimum at the boundary, it is useful to perform this perturbation in an extra collar attached to the boundary, so that the Lagrangians still intersect at the minimum of H if they did prior to the perturbation. Once this is done, we can identify

$$CF^*(L(d_1), L(d_2); rH) \cong CF^*(\phi_{rH}(L(d_1)), L(d_2)) \cong CF^*(L(d_1 - r), L(d_2)) \quad (7.9)$$

where $r \in \mathbb{Z}$ in case L , $r \in 2\mathbb{Z}$ in case C , and $r \in 3\mathbb{Z}$ in case D . Recall that the generators of the last group are identified with $B(\frac{1}{d_2 - d_1 + r}\mathbb{Z})$.

Once we are in the range $d_2 - d_1 + r > 0$, we find that as r increases, new generators are created, none are destroyed, and the generators that already exist are “compressed” toward the minimum of H . This gives rise to naive inclusion maps $i : CF^*(L(d_1), L(d_2); rH) \rightarrow CF^*(L(d_1), L(d_2); r'H)$ for $r < r'$, where $r > d_1 - d_2$. In terms of fractional integral points, this i corresponds to the map $B(\frac{1}{d_2 - d_1 + r}\mathbb{Z}) \rightarrow B(\frac{1}{d_2 - d_1 + r'}\mathbb{Z})$ which is dilation by the appropriate factor centered at the point corresponding to the minimum of H .

We can index the points of $\hat{B}(\frac{1}{d}\mathbb{Z})$ with two indices. The index $a \in \mathbb{Z}$ corresponds to the column lying at $\eta = a/d$, while the index i that indexes points within a column, and which lies in $\{0, \dots, \lfloor \frac{d-|a|}{2} \rfloor\}$ in the compact case, is now unbounded in the positive direction in case L , in the negative direction in case C , and in both directions in case D . We use the notation $q_{a,i}$ for these points.

7.3 Continuation maps and products

For $r > d_1 - d_2$, the generators of $CF^*(L(d_1), L(d_2); rH)$ all have degree 0, so there are no differentials, and we can identify these complexes with their homologies. In order to obtain the wrapped Floer cohomology, we must determine the continuation maps.

Let it be understood that we require $r \in \mathbb{Z}$ in case L , $r \in 2\mathbb{Z}$ in case C , and $r \in 3\mathbb{Z}$ in case D .

To get started, we consider the wrapped Floer cohomology of $L(d_1)$ with itself. Each complex $CF^*(L(d_1), L(d_1); rH)$ has a distinguished element e_r , sitting at the minimum of H . Under the $r \rightarrow r'$ continuation map, $e_r \mapsto e_{r'}$; e_r and $e_{r'}$ are the unique generators of minimal action in their respective complexes, and in fact their actions are equal, so the only strip is the constant map to the minimum.

At this point we bring in the product structure. In general, there is an identification between the product

$$HF^*(L_2, L_3; rH) \otimes HF^*(L_1, L_2; sH) \rightarrow HF^*(L_1, L_2; (r+s)H) \quad (7.10)$$

which counts inhomogeneous pseudoholomorphic triangles, and the product

$$HF^*(\phi_{rH}(L_2), L_3) \otimes HF^*(\phi_{(r+s)H}(L_1), \phi_{rH}(L_2)) \rightarrow HF^*(\phi_{(r+s)H}(L_1), L_2) \quad (7.11)$$

counting pseudoholomorphic triangles, which holds at the homology level. When the differentials vanish, this holds at the chain level as well. This latter product is what was computed in section 4. Tracing the isomorphisms through, we find that the

product with e_r induces the naive inclusion map on generators

$$\mu^2(e_r, \cdot) = i : HF^*(L(d_1), L(d_2); sH) \rightarrow HF^*(L(d_1), L(d_2); (r + s)H) \quad (7.12)$$

Due to the compatibility of the product with the continuation maps, and the fact that $e_r \mapsto e_{r'}$ under continuation, we find that the naive inclusion maps commute with the continuation maps:

$$\begin{array}{ccc} HF^*(L(d_1), L(d_2); sH) & \xrightarrow{i} & HF^*(L(d_1), L(d_2); (s + r)H) \\ \downarrow = & & \downarrow \text{cont.} \\ HF^*(L(d_1), L(d_2); sH) & \xrightarrow{i} & HF^*(L(d_1), L(d_2); (s + r')H) \end{array} \quad (7.13)$$

Thus the continuation map agrees with the naive inclusion map, at least on those generators which are in the image of $HF^*(L(d_1), L(d_2); sH)$. It follows that the continuation maps agree with the naive inclusion maps, at least for r large enough depending on a particular generator. Hence

$$HW^*(L(d_1), L(d_2)) = \lim_{r \rightarrow \infty} HF^*(L(d_1), L(d_2); rH), \quad (7.14)$$

where the limit is formed with respect to the continuation maps, or with respect to the naive inclusion maps, or (what is equal) the multiplications by the various elements e_r .

Spelling this out a bit more gives a precise correspondence with section 7.1. Consider the isomorphism

$$HF^*(L(d_1), L(d_1); rH) \cong HF^*(L(d_1 - r), L(d_1)) \cong H^*(\mathbb{C}\mathbb{P}^2, \mathcal{O}(r)) \quad (7.15)$$

- L : For $r \in \mathbb{Z}$, this isomorphism identifies e_r with y^r .
- C : For $r \in 2\mathbb{Z}$, this isomorphism identifies e_r with $p^{r/2}$.
- D : For $r \in 3\mathbb{Z}$, this isomorphism identifies e_r with $(yp)^{r/3}$.

Thus the directed systems computing wrapped Floer cohomology are identified with

those computing the cohomology of line bundles on the divisor complements.

We can identify the basis of $HW^*(L(d_1), L(d_2))$ with $\hat{B}(\frac{1}{d_2-d_1}\mathbb{Z})$, where $\hat{B} = \hat{B}_L, \hat{B}_C, \hat{B}_D$ is the completion of the affine manifold. The sets $B(\frac{1}{d_2-d_1+r}\mathbb{Z})$ embed in $\hat{B}(\frac{1}{d_2-d_1}\mathbb{Z})$ and this latter is their limit as $r \rightarrow \infty$; the map is dilation by $\frac{d_1-d_2+r}{d_2-d_1}$ centered at the minimum of H .

Using the products we computed in Chapter 4, we can identify this basis $\hat{B}(\frac{1}{d}\mathbb{Z})$ for $HW^*(L(0), L(d))$ with a basis of $H^*(U; \mathcal{O}(d))$.

- L : The point $q_{a,i}$ of $\hat{B}_L(\frac{1}{d}\mathbb{Z})$ corresponds to the function $x^{-a}p^i y^{d+a-2i}$ for $a \leq 0$, and $z^a p^i y^{d-a-2i}$ for $a \geq 0$. In this case $i \geq 0$ can be arbitrarily large, so the exponent of y is allowed to be negative.
- C : The point $q_{a,i}$ of $\hat{B}_C(\frac{1}{d}\mathbb{Z})$ corresponds to the function $x^{-a}p^i y^{d+a-2i}$ for $a \leq 0$, and $z^a p^i y^{d-a-2i}$ for $a \geq 0$. In this case $i \leq \lfloor \frac{d-|a|}{2} \rfloor$ can be negative, so the exponent of p is allowed to be negative, while the exponent of y is nonnegative.
- D : The point $q_{a,i}$ of $\hat{B}_D(\frac{1}{d}\mathbb{Z})$ corresponds to the function $x^{-a}p^i y^{d+a-2i}$ for $a \leq 0$, and $z^a p^i y^{d-a-2i}$ for $a \geq 0$. In this case $i \in \mathbb{Z}$, so the exponents of y and p are allowed to be negative.

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