BOLTZMANN WEIGHTS FOR THE JONES POLYNOMIAL AS A POTTS MODEL

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1. Introduction

The origin of this note is the course I taught in the Young Scholars Program at the University of Chicago in July 2019. The students were rising 11th and 12th graders, and the topic was knot theory. We discussed the Jones polynomial using the approach via Kauffman bracket. As one of the last topics, I wanted to show the students how the Jones polynomial arises from the partition function of a Potts-type model on the signed planar graph associated to the link diagram. In preparing this lecture, I used Jones' 1989 article in the Pacific Journal of Mathematics [2] and Adams' excellent *Knot Book* [1]. In the course of preparation, I discovered that both of these references contain incorrect statements about the Boltzmann weights that are meant to give rise to the Jones polynomial. While other references that I have found contain correct statements, they are sometimes less precise than I wished. For instance, they may give a set of weights, and claim that the resulting partition function agrees with the Jones polynomial "up to a factor," but this factor may depend on quantities, such as the numbers of vertices, positive edges, and negative edges in the graph, which are not invariant under any of the Reidemeister moves and which may be difficult to recognize when not explicitly given.

The aim of this note is to determine the Boltzmann weights for the Potts models whose partition function yields a regular isotopy invariant, and to write down the precise relationship to the Jones polynomial. There are eight possible models, and all of their partition functions may be expressed in terms of the Jones polynomial, the Tait number (writhe), and the number of components of the link.[1] The proof is based on the idea, emphasized in the works of Louis Kauffman (e.g., [3]), that the deletion-contraction recursion satisfied by the partition function of the Potts model is essentially the same as the Kauffman bracket recursion. My hope is that this minor contribution to scholarship may make it easier for students to appreciate this connection.

We shall use terminology and conventions that are common to the diagrammatic approach to the Jones polynomial. Appendix A serves as a reference for these conventions.

2. Potts model

The generalized Potts model is a spin model on a signed graph. A *signed graph* is a graph $G = (V, E)$ that has a $+$ or $-$ associated to each edge. In other words, a signed graph is a graph whose set of edges E is partitioned into two subsets $E = E_+ \cup E_-$, where E_+ is the set of $(+)$-edges and E_- is the set of $(-)$-edges.

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[1]More precisely, they may all be expressed in terms of the Kauffman bracket polynomial, the Tait number modulo four (which happens to be an unoriented regular isotopy invariant), and the number of components of the link modulo two.

In all graphs we allow multiple edges and loops. Technically this means there is a function from the edge set E to the set of unordered pairs of non-necessarily-distinct elements of V that sends an edge e to the pair (i, j) where i and j are the vertices at the ends of e. The loops are the edges that are sent to (i, i) for some $i \in V$. Let $N = |V|$ be the number of vertices.

Choose a natural number q that will serve as the number of possible spin states (or "colors") at each vertex. Each vertex i has a spin state s_i that can take any integer value from 1 to q. An overall state of the model is a tuple $s = (s_1, s_2, \ldots, s_N)$ of spin states at the N vertices. Choose weight functions $w_+(a, b)$ and $w_-(a, b)$ for the $(+)$-edges and $(-)$-edges respectively, where a and b denote spins. The partition function of the Potts model is then

$$Z = \sum_s \prod_{e \in E} w_\pm(s_i, s_j) = \sum_s \prod_{e \in E_+} w_+(s_i, s_j) \prod_{e \in E_-} w_-(s_i, s_j). \tag{1}$$

It is to be understood that the sum is over all q^N states s, and that in each product that has factors indexed by edges, s_i and s_j denote the spin states associated to the vertices i and j that are the ends of the edge under consideration.

Because our edges are undirected, in order for the formula for Z to make sense it is necessary that $w_\pm$ be symmetric:

$$w_+(a, b) = w_+(b, a), \tag{2}$$

$$w_-(a, b) = w_-(b, a). \tag{3}$$

Next, in order to deserve the name "Potts model," the weights should only depend on whether the two arguments are equal or different:

$$w_+(a, b) = \begin{cases} w_+^= & \text{if } a = b, \\ w_+^{\neq} & \text{if } a \neq b, \end{cases} \tag{4}$$

$$w_-(a, b) = \begin{cases} w_-^= & \text{if } a = b, \\ w_-^{\neq} & \text{if } a \neq b. \end{cases} \tag{5}$$

All told, this is a family of models on signed graphs with five parameters: q,$w_+^=$, $w_+^{\neq}$, $w_-^=$, and $w_-^{\neq}$.

Now, to each link diagram L we may associate a signed *planar* graph $G(L)$, and so we may associate a partition function $Z(L)$ by applying the formula (1) to $G(L)$. The problem is to regular isotopy invariance, which is to say invariance under Reidemeister II and III moves. Also, rather than considering $Z(L)$ directly, it is better to consider $q^{-N/2}Z(L)$, where N as always denotes the number of vertices in the graph. The reason for the factor $q^{-N/2}$ seems to be that the Reidemeister moves change the number of vertices N, and this factor is necessary whenever one compares Potts models on graphs with varying numbers of vertices. We shall also see that Reidemeister II invariance is actually impossible without this factor.

Note that there is a sign ambiguity in the factor $q^{-N/2}$ for N odd. As q is a natural number, we may take $q^{-N/2} > 0$.

Thus we pose the problem:

Problem. *For a given q, determine the weights $w_+^=$, $w_+^{\neq}$, $w_-^=$, and $w_-^{\neq}$ that make $q^{-N/2}Z(L)$ invariant under regular isotopy.*

The equations that must be satisfied are stated in [2]. There are two versions of each move because of the different possibilities of shading. One version of Reidemeister II yields the equation[2]

$$w_+(a,b)w_-(a,b) = 1, \tag{6}$$

which means that w_- is the reciprocal of w_+. The other version of Reidemeister II yields the equation

$$\sum_{x=1}^{q} w_-(a,x)w_+(x,b) = q\delta(a,b), \tag{7}$$

where $\delta(a,b)$ is the Kronecker delta. The factor of q is necessary on the right-hand side because the right-hand side involves a graph that has two fewer vertices than the graph that is involved in the left-hand side. Although this appears to be q^2 equations, there are really only two constraints depending on whether $a = b$ or $a \neq b$.

If $a = b$, then the left-hand side of (7) becomes

$$\sum_{x=1}^{q} w_-(a,x)w_+(a,x) = \sum_{x=1}^{q} 1 = q = q\delta(a,a) \tag{8}$$

by (6). Thus we do not get a new constraint. Here we remark that, if we had not considered $q^{-N/2}Z(L)$ but rather $Z(L)$ itself, we would run into an inconsistency at this point.

If $a \neq b$, then the left-hand side of (7) becomes

$$\begin{aligned} &w_-(a,a)w_+(a,b) + w_-(a,b)w_+(b,b) + \sum_{x\neq a,b} w_-(a,x)w_+(x,b) \\ &= w_-^{=}w_+^{\neq} + w_-^{\neq}w_+^{=} + (q-2) = (w_+^{=})^{-1}w_+^{\neq} + (w_+^{\neq})^{-1}w_+^{=} + (q-2) \end{aligned} \tag{9}$$

Define a new parameter t by

$$t = -(w_+^{=})^{-1}w_+^{\neq}. \tag{10}$$

Then the condition that (9) vanish becomes $-t - t^{-1} + (q-2) = 0$, or

$$q = 2 + t + t^{-1}, \tag{11}$$

which seems to be well known as the relationship between the variable t of the Jones polynomial and the number of spins in the corresponding Potts model. Note however that this relation, for given q, only determines t up to inversion. Indeed swapping w_+ with w_- swaps t with t^{-1}. Of course, we can solve this using the quadratic formula to obtain

$$t = \frac{q - 2 \pm \sqrt{q(q-4)}}{2} \tag{12}$$

The number of free parameters remaining is now reduced to a discrete choice of t or its reciprocal, and then a choice of one weight, say $v = w_+^{=}$. In terms of the parameters t and

[2]Throughout this discussion, all equations are intended to be identities, meaning that they should hold for all values of any free spin variables appearing in them.

v, we may write

$$w_+(a,b) = \begin{cases} v & \text{if } a = b, \\ -vt & \text{if } a \neq b, \end{cases} \tag{13}$$

$$w_-(a,b) = \begin{cases} v^{-1} & \text{if } a = b, \\ -v^{-1}t^{-1} & \text{if } a \neq b. \end{cases} \tag{14}$$

When considering Reidemeister III invariance, there are two versions of the move, and two possible shadings of each, but all told there are only two possible things that can happen to the graph $G(L)$, both of which are a star-triangle exchange. One is a star with two pluses and one minus becoming a triangle with two minuses and one plus. The constraint we get is

$$\sum_{x=1}^{q} w_+(a,x)w_+(b,x)w_-(c,x) = \sqrt{q}w_+(a,b)w_-(b,c)w_-(a,c), \tag{15}$$

where the factor $\sqrt{q}$ accounts for the fact that the triangle has one fewer vertex than the star. The other version of Reidemeister III gives the same constraint with $+$ and $-$ subscripts swapped; it is equivalent to (15) modulo the constraints already imposed. Now since (15) is symmetric in a and b, it represents four essentially different situations: $a = b = c$, $a = b \neq c$, $a = c \neq b$, and a, b, c all distinct.

Let us first analyze the case $a = b = c$. Then using (6), we obtain

$$\sum_{x=1}^{q} w_+(a,x) = \sqrt{q}w_-(a,a), \tag{16}$$

$$w_+^= + (q-1)w_+^{\neq} = \sqrt{q}w_-^=, \tag{17}$$

$$v + (q-1)(-vt) = \sqrt{q}v^{-1}. \tag{18}$$

We can rewrite this last equation in terms of t and v alone. The parameter t is real and positive as soon as $q \geq 4$ (and $q \geq 5$ is the generic case where $t \neq t^{-1}$). Thus we can take $t^{1/2} > 0$ when $q \geq 4$. Since we always take $\sqrt{q} > 0$, the equation

$$\sqrt{q} = t^{1/2} + t^{-1/2} \tag{19}$$

is then valid. Plugging this into (18) and simplifying gives the constraint

$$v^2 = -t^{-3/2}. \tag{20}$$

One may check that all the other cases of (15) lead to this same constraint. Thus, once t is chosen, there are two choices for v, namely

$$v = \pm i t^{-3/4}. \tag{21}$$

Theorem 2.1. *For each $q \geq 5$, there are exactly four choices of Boltzmann weights $w_+(a,b)$ and $w_-(a,b)$ such that $q^{-N/2}Z(L)$ is invariant under regular isotopy, where we take $q^{-N/2} > 0$. One such choice is*

$$w_+(a,b) = \begin{cases} it^{-3/4} & \text{if } a = b, \\ -it^{1/4} & \text{if } a \neq b, \end{cases} \tag{22}$$

$$w_-(a,b) = \begin{cases} -it^{3/4} & \text{if } a = b, \\ it^{-1/4} & \text{if } a \neq b. \end{cases} \tag{23}$$

where $t^{1/4}$ is a positive number whose fourth power is

$$t = \frac{q - 2 + \sqrt{q(q-4)}}{2}. \tag{24}$$

The other choices differ from this one by swapping t with t^{-1} and/or negating all weights.

It is possible to obtain an invariant with real weights, but in this case, we have to use the *negative square root* of q. All of the analysis is the same except that we get the relation $v^2 = t^{-3/2}$, and hence $v = \pm t^{-3/4}$.

Theorem 2.2. *For each $q \geq 5$, there are exactly four choices of Boltzmann weights $w_+(a,b)$ and $w_-(a,b)$ such that $(-\sqrt{q})^{-N} Z(L)$ is invariant under regular isotopy, where we take $\sqrt{q} > 0$. One such choice is*

$$w_+(a,b) = \begin{cases} -t^{-3/4} & \text{if } a = b, \\ t^{1/4} & \text{if } a \neq b, \end{cases} \tag{25}$$

$$w_-(a,b) = \begin{cases} -t^{3/4} & \text{if } a = b, \\ t^{-1/4} & \text{if } a \neq b. \end{cases} \tag{26}$$

where $t^{1/4}$ is a positive number whose fourth power is

$$t = \frac{q - 2 + \sqrt{q(q-4)}}{2}. \tag{27}$$

The other choices differ from this one by swapping t with t^{-1} and/or negating all weights.

We observe that the models in Theorem 2.1 differ from those in 2.2 by multiplying w_+ by $-i$ and w_- by i.

3. Deletion-contraction recursion

The partition function of the Potts model (1) satisfies a recursion relation that deletes and contracts an edge of the graph. If $e \in E$ is an edge of G, then deleting e produces a graph $\mathsf{Del}(G,e)$, and contracting e produces a graph $\mathsf{Ctr}(G,e)$. Both $\mathsf{Del}(G,e)$ and $\mathsf{Ctr}(G,e)$ have edge set equal to $E \setminus \{e\}$, while $\mathsf{Del}(G,e)$ has the same set of vertices as G, and $\mathsf{Ctr}(G,e)$ has one fewer vertex *unless* e is a loop; if e is a loop then $\mathsf{Ctr}(G,e) = \mathsf{Del}(G,e)$.

To derive the deletion-contraction recursion, first consider the case where $w_+ = w_-$ (that is, there is only one type of edge). So each edge carries weight

$$w(a,b) = \begin{cases} w^{=} & \text{if } a = b, \\ w^{\neq} & \text{if } a \neq b. \end{cases} \tag{28}$$

It is helpful to rewrite this using delta functions as

$$w(a,b) = w^{=}\delta(a,b) + w^{\neq}(1 - \delta(a,b)) = w^{\neq} + (w^{=} - w^{\neq})\delta(a,b) \tag{29}$$

Now compare the partition functions $Z(G)$ and $Z(\mathsf{Del}(G,e))$. The two models have the same set of states, and each term in $Z(G)$ has, in comparison to $Z(\mathsf{Del}(G,e))$, an extra factor of $w(s_i, s_j)$, where $i, j \in V$ are the ends of e. In view of (29), this extra factor always includes a term $w^{\neq}$, so we find

$$Z(G) = w^{\neq} Z(\mathsf{Del}(G,e)) + Y, \tag{30}$$

where Y is the correction corresponding to states where the delta function term in (29) is non-zero. Those terms are precisely those where $s_i = s_j$, and hence are in bijection with the states of $\mathsf{Ctr}(G, e)$. Thus we conclude

$$Z(G) = w^{\neq} Z(\mathsf{Del}(G, e)) + (w^{=} - w^{\neq}) Z(\mathsf{Ctr}(G, e)). \tag{31}$$

The case where there are two kinds of edges is no different, and we find

$$Z(G) = \begin{cases} w_+^{\neq} Z(\mathsf{Del}(G, e)) + (w_+^{=} - w_+^{\neq}) Z(\mathsf{Ctr}(G, e)) & \text{if } e \in E_+, \\ w_-^{\neq} Z(\mathsf{Del}(G, e)) + (w_-^{=} - w_-^{\neq}) Z(\mathsf{Ctr}(G, e)) & \text{if } e \in E_-. \end{cases} \tag{32}$$

Returning to the case of a single edge type, there are two special cases where the $Z(G)$ is directly proportional to $Z(\mathsf{Ctr}(G, e))$. One is where e is a loop based at a vertex i. In this case, we have

$$Z(G) = w^{=} Z(\mathsf{Ctr}(G, e)), \tag{33}$$

which may be seen either directly from the definition, or as a special case of deletion-contraction with $\mathsf{Del}(G, e) = \mathsf{Ctr}(G, e)$. The second is where e is an edge touching a 1-valent vertex i. In this case $\mathsf{Del}(G, e)$ may be identified with the disjoint union of $\mathsf{Ctr}(G, e)$ and i. Thus $Z(\mathsf{Del}(G, e)) = qZ(\mathsf{Ctr}(G, e))$ and we have

$$Z(G) = (w^{\neq} q + w^{=} - w^{\neq}) Z(\mathsf{Ctr}(G, e)). \tag{34}$$

The case with $(\pm)$-edges is the same, and the equations above hold with appropriate subscripts.

3.1. **Normalized form.** Our regular isotopy invariant is not Z itself but the normalized form $(\sqrt{q})^{-N} Z$ where N is the number of vertices. Let us denote by $\tilde{Z}$ this normalized version. In this section $\sqrt{q}$ should be understood as any fixed number whose square is q, in order to encompass both cases in the analysis of section 2. We must take care that $\mathsf{Ctr}(G, e)$ has one fewer edge than G except when e is a loop. Multiplying (31) by $(\sqrt{q})^{-N}$, where N is the number of edges in G, we obtain

$$(\sqrt{q})^{-N} Z(G) = w^{\neq}[(\sqrt{q})^{-N} Z(\mathsf{Del}(G, e))] + \frac{w^{=} - w^{\neq}}{\sqrt{q}} [(\sqrt{q})^{-(N-1)/2} Z(\mathsf{Ctr}(G, e))] \tag{35}$$

If e is not a loop this reads

$$\tilde{Z}(G) = w^{\neq} \tilde{Z}(\mathsf{Del}(G, e)) + \frac{w^{=} - w^{\neq}}{\sqrt{q}} \tilde{Z}(\mathsf{Ctr}(G, e)) \tag{36}$$

while if e is a loop we have

$$\tilde{Z}(G) = w^{=} \tilde{Z}(\mathsf{Ctr}(G, e)) \tag{37}$$

Also observe that (34) now reads

$$\tilde{Z}(G) = \left(w^{\neq} \sqrt{q} + \frac{w^{=} - w^{\neq}}{\sqrt{q}} \right) \tilde{Z}(\mathsf{Ctr}(G, e)) \tag{38}$$

3.2. **Deletion-contraction for regular isotopy invariants.** Now let us consider the regular isotopy invariant found in Theorem 2.2 which has real weights and $w_+(a,a) = -t^{-3/4}$. In this case we need to take the negative square root of q; for explicitness we take $\sqrt{q} > 0$ and set $\tilde{Z} = (-\sqrt{q})^{-N} Z$. Since $t^{1/4}$ is taken to be a positive real number, this makes the equation $\sqrt{q} = t^{1/2} + t^{-1/2}$ valid.

The deletion-contraction relation for this model reads

$$Z(G) = \begin{cases} t^{1/4} Z(\mathsf{Del}(G,e)) + (-t^{-3/4} - t^{1/4}) Z(\mathsf{Ctr}(G,e)) & \text{if } e \in E_+, \\ t^{-1/4} Z(\mathsf{Del}(G,e)) + (-t^{3/4} - t^{-1/4}) Z(\mathsf{Ctr}(G,e)) & \text{if } e \in E_-. \end{cases} \tag{39}$$

For the normalized form, we divide the coefficient of the second term on the right hand side by the relevant square root of q, which in this case is $-\sqrt{q} = -t^{1/2} - t^{-1/2}$. After simplification we get

$$\tilde{Z}(G) = \begin{cases} t^{1/4} \tilde{Z}(\mathsf{Del}(G,e)) + t^{-1/4} \tilde{Z}(\mathsf{Ctr}(G,e)) & \text{if } e \in E_+ \text{ and } e \text{ is not a loop}, \\ t^{-1/4} \tilde{Z}(\mathsf{Del}(G,e)) + t^{1/4} \tilde{Z}(\mathsf{Ctr}(G,e)) & \text{if } e \in E_- \text{ and } e \text{ is not a loop}. \end{cases} \tag{40}$$

4. Kauffman bracket

Let us know recall the recursive definition of the Kauffman bracket $\langle L \rangle$ of a link diagram L. For each crossing c of L, we may modify the link diagram in two possible ways, the A-split and the B-split. Let us denote these by $\mathsf{Asp}(L,c)$ and $\mathsf{Bsp}(L,c)$ respectively. The Kauffman bracket recursion is

$$\langle L \rangle = A \langle \mathsf{Asp}(L,c) \rangle + A^{-1} \langle \mathsf{Bsp}(L,c) \rangle. \tag{41}$$

Since both $\mathsf{Asp}(L,c)$ and $\mathsf{Bsp}(L,c)$ have one fewer crossing than L, this recursion allows us to reduce the number of crossings to zero, resulting in a diagram that is a disjoint union of circles (which may nevertheless be nested inside one another in the plane). The base case for the Kauffman bracket is

$$\langle \text{disjoint union of } m \text{ circles} \rangle = (-A^2 - A^{-2})^{m-1}. \tag{42}$$

4.1. **A- and B-splits in terms of graphs.** The A- and B-splits of L may be interpreted in terms of signed planar graphs. Recall that an edge of the graph carries a $+$ if the two A-regions at the corresponding crossing are shaded, and it carries a $-$ if the two B-regions at the corresponding crossing are shaded. Thus the correspondence reads

$$L \leftrightarrow G \tag{43}$$

$$c \leftrightarrow e \tag{44}$$

$$\mathsf{Asp}(L,c) \leftrightarrow \begin{cases} \mathsf{Ctr}(G,e) & \text{if } e \in E_+ \text{ and } e \text{ is not a loop}, \\ \mathsf{Del}(G,e) & \text{if } e \in E_-, \end{cases} \tag{45}$$

$$\mathsf{Bsp}(L,c) \leftrightarrow \begin{cases} \mathsf{Del}(G,e) & \text{if } e \in E_+, \\ \mathsf{Ctr}(G,e) & \text{if } e \in E_- \text{ and } e \text{ is not a loop}. \end{cases} \tag{46}$$

Let us remark on the requirement that e not be a loop in some of these cases. The correspondence between link diagrams and signed planar graphs is subject to an ambiguity when one of the shaded regions in the link diagram fails to be simply connected. For instance, consider two circles, one inside the other, with the annulus between them shaded. The corresponding graph consists of a single vertex and no edges, which is the same result as if the diagram consisted of a single circle. For this reason, the above correspondence is only valid when the shading of L has all simply connected shaded regions, and the case of $\mathsf{Ctr}(G,e)$ when e is a loop must be excluded because this operation creates an annular shaded region in the link diagram.

If G is a signed planar graph and e is an edge which is not a loop, let us define $\mathsf{Asp}(G,e)$ and $\mathsf{Bsp}(G,e)$ so as to match the operations on link diagrams. That is, define

$$\mathsf{Asp}(G,e) = \begin{cases} \mathsf{Ctr}(G,e) & \text{if } e \in E_+ \text{ and } e \text{ is not a loop,} \\ \mathsf{Del}(G,e) & \text{if } e \in E_-, \end{cases} \tag{47}$$

$$\mathsf{Bsp}(G,e) = \begin{cases} \mathsf{Del}(G,e) & \text{if } e \in E_+, \\ \mathsf{Ctr}(G,e) & \text{if } e \in E_- \text{ and } e \text{ is not a loop.} \end{cases} \tag{48}$$

This notation allows us to write the deletion-contraction recursion (40) as

$$\tilde{Z}(G) = t^{-1/4}\tilde{Z}(\mathsf{Asp}(G,e)) + t^{1/4}\tilde{Z}(\mathsf{Bsp}(G,e)) \quad \text{if } e \in E_- \cup E_+ \text{ and } e \text{ is not a loop.} \tag{49}$$

By comparing (41) and (49) we see they are essentially the same up to the change of variable $A = t^{-1/4}$.

It remains to compare the base cases for the two recursions. The base case for the Kauffman bracket is a disjoint union of circles (as a link diagram), while the base case for (49) is a disjoint union of bouquets of circles (as a signed planar graph). Consider first the split unlink U_m consisting of m unnested circles. The corresponding graph G_m consists of m vertices and no edges. Then

$$\langle U_m \rangle_{A=t^{-1/4}} = (-A^2 - A^{-2})^{m-1}|_{A=t^{-1/4}} = (-t^{1/2} - t^{-1/2})^{m-1} = (-\sqrt{q})^{m-1} \tag{50}$$

$$\tilde{Z}(G_m) = (-\sqrt{q})^{-m} Z(G_m) = (-\sqrt{q})^{-m} q^m = (-\sqrt{q})^m \tag{51}$$

This suggests that $\tilde{Z}$ differs from the Kauffman bracket by a factor of $-\sqrt{q} = -t^{1/2} - t^{-1/2} = -A^2 - A^{-2}$.

Theorem 4.1. *Let L be a link diagram shaded in such a way that all shaded regions are simply connected. Let G be the corresponding signed planar graph, and N the number of vertices of G. Let $\langle L \rangle \in \mathbb{Z}[A, A^{-1}]$ denote the Kauffman bracket, and let Z denote the partition function of the Potts model with the weights $w_+(a,b)$ and $w_-(a,b)$ as displayed in Theorem 2.2. Then*

$$\langle L \rangle_{A=t^{-1/4}} = (-\sqrt{q})^{-N-1} Z(G) = (-t^{1/2} - t^{-1/2})^{-1} \tilde{Z}(G). \tag{52}$$

5. Jones polynomial

Theorem 4.1 is a comparison between two regular isotopy invariants. To get ambient isotopy invariants we need to consider the effect of Reidemeister I moves. Denote by $\mathrm{Tait}(L)$ the *Tait number* or *writhe* of an oriented diagram L; it is the count of crossings with the same sign convention as is used to compute the linking number (which is different from the signs on the edges of G). Following Adams [1], we define

$$X(L) = (-A^{-3})^{\mathrm{Tait}(L)} \langle L \rangle \in \mathbb{Z}[A, A^{-1}] \tag{53}$$

Then the Jones polynomial V_L is

$$V_L = X(L)_{A=t^{-1/4}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}] \tag{54}$$

where $t^{1/2}$ is regarded here as a formal variable.

It is a simple matter to interpret this factor $(-A^{-3})^{\mathrm{Tait}(L)}$ in terms of the deletion-contraction recursion. The moves on graphs corresponding to Reidemeister I are the removal of a loop edge e that has nothing inside it, and removal of a 1-valent vertex together with the incident edge. Each move has two forms depending on whether the edge has a plus or minus.

Suppose e is a loop in G, and G' is the graph obtained by removing the loop. Then by (37) we have

$$\tilde{Z}(G) = w^{=}_{\pm}\tilde{Z}(G') = \begin{cases} -t^{-3/4}\tilde{Z}(G') & \text{if } e \in E_+, \\ -t^{3/4}\tilde{Z}(G') & \text{if } e \in E_-. \end{cases} \tag{55}$$

Now suppose that v is a 1-valent vertex in G, and e is the incident edge. Let G' be the graph with v and e removed. Then by (38) (minding the negative square root of q),

$$\tilde{Z}(G) = \left(-w^{\neq}_{\pm}\sqrt{q} - \frac{w^{=}_{\pm} - w^{\neq}_{\pm}}{\sqrt{q}}\right)\tilde{Z}(G') = \begin{cases} -t^{3/4}\tilde{Z}(G') & \text{if } e \in E_+, \\ -t^{-3/4}\tilde{Z}(G') & \text{if } e \in E_-. \end{cases} \tag{56}$$

The relationship to the Tait number is as follows. A loop $e \in E_+$ corresponds to a crossing that contributes $+1$ to $\mathrm{Tait}(L)$, while a loop $e \in E_-$ contributes -1 to $\mathrm{Tait}(L)$. On the other hand, if $e \in E_+$ is an edge touching a 1-valent vertex, then e contributes -1 to $\mathrm{Tait}(L)$, while if $e \in E_-$ touches 1-valent vertex it contributes $+1$ to $\mathrm{Tait}(L)$. If we denote by $\mathrm{Tait}(e)$ the contribution of the edge e to the Tait number, we find that in all four cases (($\pm$)-edges, loops and 1-valent vertices)

$$\tilde{Z}(G) = (-t^{-3/4})^{\mathrm{Tait}(e)}\tilde{Z}(G') \tag{57}$$

This shows that the combination

$$\hat{Z}(G) = (-t^{3/4})^{\mathrm{Tait}(L)}\tilde{Z}(G) \tag{58}$$

is invariant under Reidemeister I moves as well.

Theorem 5.1. *Let L be an oriented link diagram shaded in such a way that all shaded regions are simply connected. Let G be the corresponding signed planar graph, and N the number of vertices of G. Let $V_L \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ be the Jones polynomial of L, and let Z denote the partition function of the Potts model with the weights displayed in Theorem 2.2. Then*

$$Z(G) = (-t^{-3/4})^{\mathrm{Tait}(L)}(-t^{1/2} - t^{-1/2})^{N+1}V_L(t), \tag{59}$$

$$\tilde{Z}(G) = (-t^{-3/4})^{\mathrm{Tait}(L)}(-t^{1/2} - t^{-1/2})V_L(t), \tag{60}$$

$$\hat{Z}(G) = (-t^{1/2} - t^{-1/2})V_L(t). \tag{61}$$

6. Comparison of other models

Theorems 2.1 and 2.2 show that for each $q \geq 5$, the q-spin Potts model yields eight slightly different regular isotopy invariants, and we now wish to compare them. Since each model has a different weight for $w_+(a,b)$, we can index the models by this choice. The main model we have been considering so far is $Z_{-t^{-3/4}}$ (which is the subject of Theorem 5.1), but we also have

$$Z_{t^{-3/4}}, Z_{-t^{3/4}}, Z_{t^{3/4}}, Z_{it^{-3/4}}, Z_{-it^{-3/4}}, Z_{it^{3/4}}, Z_{-it^{3/4}} \tag{62}$$

The normalized versions of these partition functions are denoted with a tilde, and they are obtained by multiplying by $(\pm\sqrt{q})^{-N}$, where $\sqrt{q} > 0$, we take the negative sign when the weights are real and the plus sign when the weights are imaginary, and N is the number of vertices in the graph.

The effect of negating all weights is to multiply Z by $(-1)^{|E|}$. Observe that $|E|$ is the number of crossings in the link diagram, and the number of crossings is congruent modulo two to the Tait number. The effect of swapping t with t^{-1} in the weights is to perform the same transformation on the partition function. Thus

$$Z_{t^{-3/4}} = (-1)^{\mathrm{Tait}} Z_{-t^{-3/4}}, \tag{63}$$

$$Z_{-t^{3/4}} = Z_{-t^{-3/4}}[t \to t^{-1}], \tag{64}$$

$$Z_{t^{3/4}} = (-1)^{\mathrm{Tait}} Z_{-t^{-3/4}}[t \to t^{-1}]. \tag{65}$$

In order to obtain $Z_{it^{-3/4}}$ from $Z_{-t^{-3/4}}$, we multiply w_+ by $-i$ and w_- by i. This has the effect of multiplying Z by $(-i)^{|E_+|}(i)^{|E_-|} = (i)^{|E_-|-|E_+|}$. Thus

$$Z_{it^{-3/4}} = (i)^{|E_-|-|E_+|} Z_{-t^{-3/4}}. \tag{66}$$

At first glance this may appear to be a problem because the residue class of $|E_-| - |E_+|$ modulo 4 is not invariant under Reidemeister III moves. However, we must recall that regular isotopy invariants are obtained from different normalizations on the two sides, and so

$$\tilde{Z}_{it^{-3/4}} = (-1)^N (i)^{|E_-|-|E_+|} \tilde{Z}_{-t^{-3/4}}. \tag{67}$$

This implies that the factor $(-1)^N (i)^{|E_-|-|E_+|}$ is a regular isotopy invariant, which is easy to check directly. As one might expect, this factor is related to the Tait number.

To see this, consider the combination $\xi = (-1)^N (i)^{|E_-|-|E_+|} (i)^{\mathrm{Tait}}$. We already know this is a regular isotopy invariant, but it is also invariant under Reidemeister I moves. Indeed, a loop $e \in E_+$ contributes $+1$ to $|E_+|$ and to Tait, so ξ is invariant under removal of this loop. A loop $e \in E_-$ contributes $+1$ to $|E_-|$ and to Tait, so removing it also does not change ξ. The configuration consisting of an edge $e \in E_+$ connected to a 1-valent vertex contributes $+1$ to N, $+1$ to $|E_+|$ and -1 to Tait, and so contributes a factor of $(-1)(i)^{-1}(i)^{-1} = 1$ to ξ. The remaining case is similar to this one. This proves that ξ is an oriented link invariant, but even more is true, since ξ is invariant under changing a crossing. The operation of changing a crossing changes $|E_-| - |E_+|$ by ± 2, and it changes Tait by ± 2 as well. Since it is only the sum $|E_-| - |E_+| + \mathrm{Tait}$ modulo four that matters, these changes cancel out. This proves that ξ can only depend on the number of components of L, and by computing it for the split unlink U_m, we find that

$$(-1)^N (i)^{|E_-|-|E_+|} (i)^{\mathrm{Tait}} = (-1)^{\ell} \tag{68}$$

where ℓ is the number of components of the link in question.

Thus, for the models with imaginary weights, we have

$$\tilde{Z}_{it^{-3/4}} = (-1)^{\ell} (-i)^{\mathrm{Tait}} \tilde{Z}_{-t^{-3/4}}, \tag{69}$$

$$\tilde{Z}_{-it^{-3/4}} = (-1)^{\ell} (i)^{\mathrm{Tait}} \tilde{Z}_{-t^{-3/4}}, \tag{70}$$

$$\tilde{Z}_{it^{3/4}} = (-1)^{\ell} (-i)^{\mathrm{Tait}} \tilde{Z}_{-t^{-3/4}}[t \to t^{-1}], \tag{71}$$

$$\tilde{Z}_{-it^{3/4}} = (-1)^{\ell} (i)^{\mathrm{Tait}} \tilde{Z}_{-t^{-3/4}}[t \to t^{-1}], \tag{72}$$

where ℓ is the number of components of the link.

6.1. **Models that do not respect star-triangle exchange.** In the literature [1, 2], one may find choices of weights for the Potts model for which the star-triangle exchange relation (15) is not satisfied. Hence the resulting normalized partition function is not invariant under Reidemeister III moves. But even in these cases, the Jones polynomial can still be found in the partition function.

Recalling the discussion of section 2, the most general form for weights that satisfy the Reidemeister II constraints (6) and (7) is

$$w_+(a,b) = \begin{cases} -\lambda t^{-3/4} & \text{if } a = b, \\ \lambda t^{1/4} & \text{if } a \neq b, \end{cases} \tag{73}$$

$$w_-(a,b) = \begin{cases} -\lambda^{-1} t^{3/4} & \text{if } a = b, \\ \lambda^{-1} t^{-1/4} & \text{if } a \neq b, \end{cases} \tag{74}$$

where t is a number that satisfies $q = 2+t+t^{-1}$, and λ can be any non-zero complex number. For concreteness, take t as in Theorem 2.2. Then these weights differ from the ones displayed in Theorem 2.2 by multiplying w_+ by λ and w_- by λ^{-1}. Thus the partition function satisfies

$$Z_{-\lambda t^{-3/4}} = \lambda^{|E_+|-|E_-|} Z_{-t^{-3/4}}. \tag{75}$$

Hence, under the hypotheses of Theorem 5.1, we have

$$Z_{-\lambda t^{-3/4}}(G) = \lambda^{|E_+|-|E_-|}(-t^{-3/4})^{\mathrm{Tait}(L)}(-t^{1/2} - t^{-1/2})^{N+1} V_L(t). \tag{76}$$

In Jones' PJM article [2], he chooses $w_+^= = 1$ and $w_+^{\neq} = -t^{-1}$. This corresponds to taking $\lambda = -t^{3/4}$ and then swapping t with t^{-1}. With this choice, the partition function is

$$Z_{\mathrm{PJM}}(G) = Z_1(G)[t \to t^{-1}] = (-t^{-3/4})^{|E_+|-|E_-|-\mathrm{Tait}(L)}(-t^{1/2} - t^{-1/2})^{N+1} V_L(t^{-1}). \tag{77}$$

Appendix A. Conventions

Here we summarize our conventions, all of which are standard.

A.1. **A and B.** At a crossing, we have an overstrand and an understrand. Locally, these strands divide the plane into four quadrants. If we rotate the overstrand counterclockwise towards the understrand, it sweeps across two quadrants, which we label A. The other two quadrants are labeled B.

Given a crossing c of a link diagram L, the A*-split* of L at c is the modification that removes c and connects the two A-quadrants. The B*-split* of L at c is the modification that removes c and connects the two B-quadrants.

A.2. **Shading.** Given a link diagram L, a *shading* of L is a shading of the regions between the strands of L, in such a way that a shaded region is adjacent only to unshaded regions and vice versa. In other words, if we regard L as a 4-regular planar graph, a shading is a proper 2-coloring of the dual graph of L, and the names for the two colors are "shaded" and "unshaded." We assume the infinite region is unshaded unless otherwise specified.

A.3. **Medial graphs.** From a link diagram L with shading, we construct a *signed planar graph* G corresponding to L. To do this, we put a vertex in each shaded region, and an edge connecting two vertices if the regions meet at a crossing in a diagonally opposite fashion. The edge is decorated by $+$ or $-$. It receives a $+$ if the two A-quadrants are shaded at the crossing where the regions meet, and it receives a $-$ if the two B-quadrants are shaded there.

Conversely, from a signed planar graph G we may construct a link diagram. To do this we put a 4-valent vertex at the midpoint of each edge of G, and we draw strands so as to connect edges that are adjacent around the faces of G. This is also known as the *medial graph* construction. The result is a 4-regular planar graph with a shading, and we use the $\pm$-decorations on the edges to determine the over/understands at each crossing.

The correspondence between link diagrams and signed planar graphs is bijective if we assume the diagrams and graphs are connected, or more generally if we assume that all regions in the link diagram are simply connected. There are some subtleties without this assumption.

A.4. **Signs and Tait number.** In an oriented link, each crossing has another *sign*, which is unrelated to the signs on the edges of the corresponding graph. This sign is the one used to define the linking number. Our convention is that, when we rotate the crossing so that the two oriented strands are pointing northeast and northwest, then the crossing is positive if the vector pointing north lies an an A-quadrant, and the crossing is negative if this vector lies in a B-quadrant. The count of all crossings with these signs is called the *Tait number*, also known as the *writhe* or *twist number*.

A.5. **Ambient and regular isotopy.** The equivalence relation on link diagrams generated by planar isotopy and the three Reidemeister moves is called *ambient isotopy*. It is the usual equivalence relation on link diagrams. The equivalence relation on link diagrams generated by planar isotopy and Reidemeister moves II and III only is called *regular isotopy*. For example, the Tait number is invariant under regular isotopy but not ambient isotopy.

References

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