Lecture 38: One parameter deformation theory

The general framework of one-parameter deformation theory involves a DG Lie algebra $(\mathfrak{g}, \delta, [\,,\,])$

$(CC^{\cdot}(A,A)[-1], \delta, [\,,\,])$ is an instance.

Set $\Lambda_{\mathbb{N}} = \mathbb{C}[[q]]$. Then $\mathfrak{g} \otimes (q\Lambda_{\mathbb{N}})$ is "pro-nilpotent"

The set of Maurer–Cartan elements is

$$\mathcal{U}_q(\mathfrak{g}) = \left\{ \alpha_q \in \mathfrak{g}^1 \otimes q\Lambda_{\mathbb{N}} \;\middle|\; \delta\alpha_q + \tfrac{1}{2}[\alpha_q, \alpha_q] = 0 \right\}$$

Writing $\alpha_q = \alpha_{q,1}\, q + \alpha_{q,2}\, q^2 + \cdots$, the leading term defines a class $[\alpha_{q,1}] \in H^1(\mathfrak{g})$, the deformation class.

Now $\mathfrak{g}^0 \otimes q\Lambda_{\mathbb{N}}$ is a pro-nilpotent ordinary Lie algebra, so it integrates to a Lie group over $\Lambda_{\mathbb{N}}$

$$\mathcal{G}_q(\mathfrak{g}) = \exp(\mathfrak{g}^0 \otimes q\Lambda_{\mathbb{N}})$$

$\mathcal{G}_q(\mathfrak{g})$ acts on $\mathcal{U}_q(\mathfrak{g})$

There is also an action of $\mathrm{End}(\Lambda_{\mathbb{N}})$ by reparametrization.

The basic lemma we need is:

Lemma: Suppose $H^1(\mathfrak{g}) \cong \mathbb{C}$, and let $\alpha_q \in \mathcal{U}_q(\mathfrak{g})$ be such that $[\alpha_{q,1}] \neq 0 \in H^1(\mathfrak{g})$. Then any other element of $\mathcal{U}_q(\mathfrak{g})$ can be obtained from α_q by a combination of the gauge group action and reparametrization.

We have used similar results already, so let's see the proof.

Proof: Let $\beta_q \in \mathcal{U}_q(\mathfrak{g})$

Consider the statement for $d \geq 1$:

$\exists\, \psi^{(d)} \in \mathrm{End}(\Lambda_W)$ and $\beta_q^{(d)} \in \mathcal{U}_q(\mathfrak{g})$ such that

(i) $\beta_q^{(d)}$ is gauge equivalent to β_q

(ii) $\beta_q^{(d)} - (\psi^{(d)})^* \alpha_q \in \mathcal{O}(q^d)$

We prove this statement by induction on d. The case $d=1$ is clear: take $\beta_q^{(1)} = \beta_q$ $\psi^{(1)} = \mathrm{Id}$.

Suppose the statement is true for given d. Set $\varepsilon_q = \beta_q^{(d)} - (\psi^{(d)})^* \alpha_q$

Now (dropping decorations)

$$0 = \delta\beta + \tfrac{1}{2}[\beta,\beta] = \delta(\psi^*\alpha + \varepsilon) + \tfrac{1}{2}[\psi^*\alpha + \varepsilon, \psi^*\alpha + \varepsilon]$$

$$= \delta\psi^*\alpha + \delta\varepsilon + \tfrac{1}{2}[\psi^*\alpha, \psi^*\alpha] + [\psi^*\alpha, \varepsilon] + \tfrac{1}{2}[\varepsilon,\varepsilon]$$

$$= \delta\varepsilon + \tfrac{1}{2}[2\psi^*\alpha + \varepsilon, \varepsilon] = \delta\varepsilon + \tfrac{1}{2}[\beta + \psi^*\alpha, \varepsilon]$$

So $$\delta\varepsilon_q = -\tfrac{1}{2}[\beta_q^{(d)} + (\psi^{(d)})^*\alpha_q, \varepsilon_q] \in \mathcal{O}(q^{d+1})$$

while $\varepsilon_q \in \mathcal{O}(q^d)$. So the leading term $\varepsilon_{q,d}$ is closed, and represents a class $[\varepsilon_{q,d}] \in H^1(\mathfrak{g})$. Thus

$$[\varepsilon_{q,d}] = c[\alpha_{q,1}] \quad \text{for some } c \in \mathbb{C}$$

and

$$\varepsilon_{q,d} = c\alpha_{q,1} + \delta b \quad \text{for some } b \in \mathfrak{g}^0$$

So $\beta_q^{(d)} = (\psi^{(d)})^*\alpha_q + \varepsilon_q = (\psi^{(d)})^*\alpha_q + c\alpha_{q,1}q^d + \delta b\, q^d + \mathcal{O}(q^{d+1})$

Then define $\psi^{(d+1)}(q) = \psi^{(d)}(q) + cq^d$

$$\beta_q^{(d+1)} = \exp(-q^d b)(\beta_q^{(d)})$$

and we obtain $\beta_q^{(d+1)} - (\psi^{(d+1)})^* \alpha_q \in \mathcal{O}(q^{d+1})$
which is the statement for $d+1$.

Since the parameter change and gauge transformation are identity to order q^d at the d-th step, they converge q-adically to $\psi^{(\infty)} \in \mathrm{End}(\Lambda_N)$ and $\beta_q^{(\infty)} \in \mathcal{U}_q(\mathfrak{g})$
such that
$(\psi^{(\infty)})^* \alpha_q = \beta_q^{(\infty)}$ and $\beta_q^{(\infty)}$ is gauge-equiv to β_q. $\blacksquare$

The application we need is: let $\mathcal{A}$ be an A_∞-algebra with $\mu^1_{\mathcal{A}} \equiv 0$
let $CC(\mathcal{A},\mathcal{A})^{\leq 0}$ denote the subcomplex of cochains
$\tau : \mathcal{A}^{\otimes s} \to \mathcal{A}[t]$ with $t \leq 0$
(need to avoid deformations that turn on μ^0 or μ^1)
consider $\mathcal{U}_q(\mathcal{A})^{\leq 0}$, the 1-parameter deformations with $\mu^1_{\mathcal{A}_q} \equiv 0$.

<u>Lemma</u>: Suppose $HH^2(\mathcal{A},\mathcal{A}) \cong \mathbb{C}$, and $\mathcal{A}_q \in \mathcal{U}_q(\mathcal{A})^{\leq 0}$ is such that $[\mu^*_{q,1}] \neq 0 \in HH^2(\mathcal{A},\mathcal{A})$. Then any other $\mathcal{B}_q \in \mathcal{U}_q(\mathcal{A})^{\leq 0}$ is gauge equivalent to $\psi^* \mathcal{A}_q$ for some $\psi \in \mathrm{End}(\Lambda_N)$.