Lecture 36  Start the quartic surface.

Let $X_0$ be a quartic surface, equipped with its standard symplectic structure (restriction of the Fubini-Study Kähler form on $\mathbb{CP}^3$).

Up to symplectomorphism, $X_0$ does not depend on the equation that defines it, so we can take, for instance $X_0 = \frac{3}{2} x_0^2 + x_1^4 + x_2^4 + x_3^4 = 0$.

Similarly, we will use a version of the Fukaya category that is defined over

$$\Lambda_0 = \left\{ f(q) = \sum_m a_m q^m \quad \mid \begin{array}{l} a_m \in \mathbb{C}, \ m \text{ runs over} \\ (d-2) \leq \Omega \text{ for some } d \text{ depending on } f \\ a_m = 0 \text{ for } m < 0 \end{array} \right\}$$

The category $\text{Fuk}(X_0)$ is defined over $\Lambda_0$, provided we restrict to "rational" Lagrangians.

The mirror variety to $X_0$ is expected, from physics arguments, to be $\mathbb{Z}_q^*$, constructed as follows.

Take $Y_2 = \frac{3}{2} y_0 y_1 y_2 y_3 + \frac{3}{2} (y_0^4 + y_1^4 + y_2^4 + y_3^4) = 0$.

Set $\Gamma_6 = \left\{ [\text{diag}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)] \mid \alpha_2 = 1, \alpha_0, \alpha_1, \alpha_3 = 1 \right\} \subseteq \text{PSL}(4, \mathbb{C})$

($\Gamma_6 \cong \mathbb{Z}_4 \times \mathbb{Z}_4$) this acts on $Y_2$, but

$$\Lambda_0 \to R : q \mapsto \bar{q}$$
Set \( Z^*_q = \hat{Y}_q/\Gamma_0 \), where \( \sim \) denotes the minimal resolution of the (singular) quotient surface.

Seidel's theorem for the quartic surface is then

\[
\text{Fuk}(X_0)^{\text{per}} \cong \hat{\psi}^* \text{D}^b\text{Coh}(Z^*_q)
\]

where \( \hat{\psi} \) is an automorphism of \( \Lambda_q/C \) that is used to twist the \( \Lambda_q \)-linear structure. \( \psi \) is known as a "mirror map."

The proof has steps that are somewhat analogous to what we went through for the torus, but they are more complicated. On the symplectic side, there are several new ingredients to handle this higher-dimensional case.

One thing common to both is a detailed study of the deformation theory of a quiver algebra \( R_q^4 \) with 64 vertices. Let us approach this from the algebraic side.

\[
Z^*_q = \hat{Y}_q/\Gamma_0, \quad Y^*_q \subseteq P^3_{\Lambda_q}
\]

Bernstein quiver for \( P^3 \): Let \( V \) be a 4-dim vector space, \( P^3 = \text{PC}(V) \).

This case for concreteness, the objects \( F_k = \Sigma^{4-k}(4-k)[4-k] \) for \( k = 1,2,3,4 \).

The resulting quiver has 4 vertices; call it \( C_4 \).

\[
\text{Hom}_{C_4}(x_i, x_k) = \begin{cases} x^{k-i}V & j \leq k \\ 0 & \text{otherwise} \end{cases}
\]
It has the natural grading of the exterior algebra and amputatin comes from wedge product.

Beilinson's theorem may be stated as \( \text{D}^b \text{Coh}(\mathbb{P}^3) \cong (C_4)^{\text{perf}} \).

Next we pass to a quadric hypersurface \( Y_q \).

We can restrict \( F_k \) to \( Y_q \), and we get a quiver with the same vertices but more arrows. This is because Serre duality and \( \omega_{Y_q} \otimes \text{triv} \) implies that then spaces on \( Y_q \) are self-dual up to shift by \( \text{dim} Y_q - 2 \).

The result is a category \( C_4 \) with 4 objects \( X_1, X_2, X_3, X_4 \)

\[
\text{Hom}_{C_4}(X_j, X_k) = \begin{cases} 
\Lambda^{k-j} V & j < k \\
\Lambda^0 V \oplus (\Lambda^4 V)[2] & j = k \\
(\Lambda^{k-j+4} V)[2] & j > k 
\end{cases}
\]

We denote by \( Q_4 \) the total morphism algebra of \( C_4 \).

\( Y_q \) corresponds to some \( A_{\text{co}} \)-structure on this algebra, that we must figure out.

Next, we must pass from \( Y_q \) to \( Z_q \approx Y_q / \Gamma_6 \).

It turns out that in the realm of derived categories, this is easier than you might think.

A theorem of Kapranov-Vasserot implies

\[
\text{D}^b \text{Coh}(Z_q) \cong \text{D}^b \text{Coh}_{\Pi_6}(Y_q),
\]

where

the RHS is \( \Pi_6 \)-equivariant coherent sheaves.
In terms of quiver algebras, the equivariant structure corresponds to taking semi-direct product.

Now $GL(V)$ naturally acts on $Q_4$, and the action of $\Gamma_6 \leq PSL(V)$ can be lifted to an action of $\Gamma_6$ on $Q_4$.

We then consider $Q_{64} = Q_4 \rtimes \Gamma_6$. A certain $A^\infty$-structure on this algebra recovers $D^b Coh(\mathbb{Z}^6)$.

For a $C^*$-algebra $A$ and a group $\Gamma$ and a homomorphism $\varphi : \Gamma \to Aut(A)$, the semi-direct product is

\[ A \rtimes \Gamma = A \otimes C^* \Gamma \] with multiplication

\[ (a \otimes g)(a' \otimes g') = a \varphi(g)(a') \otimes gg' \]

It is also called "semigroup ring" or "smash product".

The algebra $Q_{64}$ can be regarded as a category with 64 objects. We need to understand what $A^\infty$-structure on $Q_{64}$ can look like.

We also need to find 64 objects in $Fuk(X_0)$ that correspond to some interesting symplectic topology here!