Recall the algebra $Q$: \[
\begin{array}{ccc}
de^0 & \circ & \deg^0 \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\] 
Let $e_1 \in \Lambda^0(V)$ and $e_2$ be the idempotents at the vertices. $Q$ is an algebra over $R_2 = \text{Re}_1 \times \text{Re}_2$ (ground field).

The Hochschild cohomology of $Q$ can be calculated:

We need

\[ (*) \quad HH^d(Q,Q[2-d]) = \begin{cases} 
0 & d = 3 \\
\text{Sym}^4(V) & d = 4 \\
0 & d \geq 5 
\end{cases} \]

\[ (**) \quad HH^d(Q,Q[3-d]) = 0 \text{ for } d \geq 7 \]

Remarks on these calculations: These use the theory of Koszul algebras. One can observe that $Q$ is isomorphic as an algebra to $\Lambda^*(V) \times \mathbb{Z}_2$ where the nontrivial $\tau \in \mathbb{Z}_2$ acts as $-1$ on $V$.

(This isomorphism is not compatible with the grading, so cave is needed.)

This algebra is Koszul, and its Koszul dual is $\text{Sym}^*(V^*) \times \mathbb{Z}_2$. (Compare the classical Koszul duality of exterior and symmetric algebras.)

The nontrivial $HH^4(Q,Q[-2]) \cong \text{Sym}^4(V^*)$ corresponds under Koszul duality to the degree 4 part of the center of $\text{Sym}^*(V^*) \times \mathbb{Z}_2$. 
So consider an $A_{\infty}$-structure $(\mu^d)_{d \geq 1}$ on $Q$ starting with that respects the grading and the $R_2$-algebra structure.

Because $HH^3(Q, Q[-1]) = 0$, after a gauge transformation we may assume $\mu^3 = 0$.

As part of $\mu^4$, we have maps

$$\mu^4_{[1212]}: e_1Qe_2 \otimes e_2Qe_1 \otimes e_1Qe_2 \otimes e_2Qe_1 \cong V^\otimes 4 \rightarrow Re_1 \cong R$$

$$\mu^4_{[2112]}: e_2Qe_1 \otimes e_1Qe_2 \otimes e_1Qe_2 \otimes e_2Qe_1 \cong V^\otimes 4 \rightarrow Re_2 \cong R$$

Using $\mu^3 = 0$ and the $A_{\infty}$-equations, one can see that these two functions are equal:

$$p(v) = \mu^4_{[1212]}(v, v, v, v) = \mu^4_{[2112]}(v, v, v, v), \quad p \in \text{Sym}^4(V)$$

In fact, under the isomorphism

$$HH^4(Q, Q[-2]) \cong \text{Sym}^4(V)$$

$$[\mu^4] \leftrightarrow p$$

Since $HH^d(Q, Q[-d]) = 0$ for $d > 5$, this class $p = [\mu^4]$ determines the $A_{\infty}$-structure up to gauge equivalence.

The fact $HH^d(Q, Q[3-d])$, $d > 7$, is used to show that an $A_{\infty}$-structure exists for any given $p$.

**Definition.** Let $Q_p$ denote $Q$ with an $A_{\infty}$-structure whose class $[\mu^4] = p$. 

Now consider the two-torus

\[ Q = \bigoplus_{i,j=1}^2 \text{HF}(L_i,L_j) \]

There is a nontrivial \( \omega \)-structure on \( Q \) determined by the \( \text{pseudo-holomorphic maps in } T \).

By the classification theory, this is determined by a \( p \in \text{Sym}^4(V) \). What \( p \) is it???

Seidel finds the answer, and it's not particularly simple:

Recall notation: \( R = \{ u = \sum_{k=0}^\infty c_k t_k^{m_k} \mid c_k \in \mathbb{C}, m_k \in \mathbb{R}, \lim_{k \to \infty} m_k = +\infty \} \)

Novikov field

\[ F = \{ f(t) = \sum_{k=0}^\infty c_k t_k^{m_k} t_k^{\mu_k} \mid c_k \in \mathbb{C}, m_k \in \mathbb{R}, \mu_k \in \mathbb{Z}, \lim_{k \to \infty} m_k + \mu_k = +\infty \} \cap \mathbb{A} \mathbb{C} \mathbb{R} \]

\[ \Theta_{n,k}(t) \in F, \quad \Theta_{n,k}(t) = \sum_{i \in \mathbb{N}^{2+k}} t_i t_i^* \]

Define the \underline{unit torsion polynomial} \( p \in \mathbb{R}[V_1,V_2] \)

\[ p(V_1,V_2) = c \left( \Theta_{2,2}(t^{1/2})^2 V_2^2 - \Theta_{2,1}(t^{1/2})^2 V_1^2 \right) \]

\[ \cdot \left( \Theta_{2,2}(1)^2 V_2^2 - \Theta_{2,1}(1)^2 V_1^2 \right) \]

where \( c = -t^{1/4} \Theta_{2,2}(-1)^{-2} \Theta_{2,1}(-1)^{-2} \Theta_{4,3}(1)^2 \)

\[ \Theta_{4,3}(1) = \left. \frac{d^3}{dt^3} \left. \frac{d}{dt} \left( \frac{1}{t} \right) \right|_{t=1} \]
This is the unique (up to scaling) homogeneous quartic polynomial that vanishes at the four points:

\[ \Theta_{2,2}(\pm t^{1/2}): \Theta_{2,1}(\pm t^{1/2}) \] , \[ \Theta_{2,2}(\pm 1): \Theta_{2,1}(\pm 1) \]

in \( \mathbb{P}^4_{\mathbb{R}} \cong \mathbb{P}(V) \)

On the algebraic side, recall the double cover:

\[ Y_p \rightarrow \mathbb{P}(V) \quad Y_p = \{ y^2 = p(v) \} \]

\[ E_i = \mathcal{O}_{\mathbb{P}(V)} \quad E_2 = \mathcal{O}_{\mathbb{P}(V)}(d) \oplus \Lambda^2(V) \]

\[ \mathcal{Q} = \bigoplus_{i,j=1}^{2} \text{Ext}^2_{Y_p}(\pi^*E_i, \pi^*E_j) \]

This algebra also inherits an \( A_{\infty} \)-structure from a DG-enhancement of the derived category \( \mathcal{D}^b(\text{Coh } Y_p) \).

It is indeed equivalent to \( Q_p \), where \( \text{peSym}^4(V) \) is the polynomial defining \( Y_p \).