We have calculated the graded algebra \( Q \) associated to our pair of Lagrangians in \( \mathcal{T} \).

We still need to figure out the \( A_{\infty} \)-structure. This is difficult to calculate directly since we have to perturb the Lagrangians several times. Another approach is to try to find invariants of \( A_{\infty} \) structures on \( Q \) that classify them.

**General theory:** Let \( A \) be a graded algebra over a field \( K \) (char 0).

Let \( \mathcal{U}(A) \) be the set of \( A_{\infty} \)-structures on \( A \), i.e., \( \{ \mu^d: A^d \to A[2-1] \} \) such that

\[
\mu^1_A = 0 \quad \mu^2_A(a_2, a_1) = (-1)^{|a_1|} a_2 a_1
\]

\[\text{(\( A_{\infty} \)-equations)} \sum_{r,s} (-1)^{s} \mu^r_{A} G_A^{sr}(a_d, \ldots, a_{d-s+1}), \ldots, G_A^{s_t}(a_s, \ldots, a_1) = 0 \]

\[* = |a_1| + \cdots + |a_s| - i\]

This means that \( H(A) = A \) as graded algebras.

Recall that an \( A_{\infty} \)-homomorphism \( G: A \to B \) is a sequence of maps

\((G_1, G_2, \ldots) \quad G^d: A^d \to B [1-d]\)

such that

\[
\sum_{r,s} \sum_{s_1, \ldots, s_r} \mu^r_B (G^{s_r}(a_d, \ldots, a_{d-s+r+1}), \ldots, G^{s_t}(a_s, \ldots, a_1)) = \sum (-1)^{s} G^{d-r+t+1}(a_d, \ldots, a_{r+t+1}, \mu^m_A(a_{r+t}, \ldots, a_1), q_n, \ldots, q_1)
\]
Two elements \( A, A' \in U(A) \) are equivalent if there is an \( A_{\infty} \)-morphism \( G : A \to A' \) such that \( G^1 = \text{Id}_A \).

In this case, \( A' \) is determined by \( A \) and \( G \), for instance

\[
\begin{align*}
\mu^3_{A'}(a_3, a_2, a_1) \\
= \mu^3_A(a_3, a_2, a_1) + G^2(a_3, \mu^2_A(a_2, a_1)) \\
+ (-1)^{|a_1| - 1} G^2(\mu^2_A(a_3, a_2), a_1) - \mu^2_A(a_3, G^2(a_2, a_1)) \\
- \mu^2_A(G^2(a_3, a_2), a_1)
\end{align*}
\]

And there are no constraints on what \( G \) can be.

This means that the set

\[
\mathcal{G}(A) = \{ (G^1 = \text{Id}_A, G^2, G^3, \ldots) \mid G^d : A^d \to A[1-d] \}
\]

acts on \( U(A) \).

\( \mathcal{G}(A) \) is called the **Gauge group**, the group law is

\[
(G \circ F)^d(a_d, \ldots, a_1) = \sum_{r, s, \ldots} \sum_{r, s, \ldots} G^r(F^r(a_s, \ldots), \ldots, F^s(a_r, \ldots, a_1))
\]

\( U(A)/\mathcal{G}(A) \) is the moduli space of \( A_{\infty} \)-structures extending the given product.

There is also an action of the monoid \( (\mathbb{K}, 1) \) on \( U(A)/\mathcal{G}(A) \):

- multiply \( \mu^d \) by \( \varepsilon^{d-2} \) for \( \varepsilon \in \mathbb{K} \).

The fixed point (\( \mu^d = 0 \) for \( d \geq 3 \)) is the **formal** \( A_{\infty} \)-structure.

We wish to understand the "tangent space" at this point.
This is described in terms of the Hochschild cohomology of $A$.

**Hochschild cohomology**: Given an associative algebra $A$, we can consider the category of $A$-bimodules.

An $A$-bimodule $M$ has a left $A$-action and a right $A$-action which commute. This is the same as a left $A \otimes A^{op}$-action.

We define

$$HH^i(A, M) = \text{Ext}^i_{A \otimes A^{op}}(A, M)$$

In particular, taking $M = A$ we get $HH^i(A, A)$.

There is a complex computing this with cochains

$$CC^i(A, M) = \text{Hom}_{K}(A^{\otimes i}, M)$$

$$(d \psi)(x_0, \ldots, x_n) = x_0 \psi(x_0, \ldots, x_n) + \sum_{i=1}^{n} (-1)^i \psi(x_0, \ldots, x_{i-1}, x_i \cdot x_{i+1}, \ldots, x_n) + (-1)^n \psi(x_0, \ldots, x_{n-1}, x_n)$$

for $\psi \in \text{Hom}_{K}(A^{\otimes n}, M)$.

When $A$ is a graded algebra, we take every thing in the graded sense. In this case $A[j]$ is a different bimodule, we twist the right action:

$$m \circ \alpha = (-1)^{j \cdot \text{deg} \alpha} m a$$

and we get a bigraded Hochschild cohomology

$$HH^i(A, A[j]) = \text{Ext}^i_{A \otimes A^{op}}(A, A[j])$$

A cochain here is $\psi \in CC^i(A, A[j]) = \text{Hom}_{K}(A^{\otimes d}, A[j])$.

Observe that the $\mu^d$ maps of an $A_\infty$-structure lie in

$$\mu^d_A \in \text{Hom}_{K}(A^d, A[2-d]) = CC^d(A, A[2-d]).$$
Proposition 4: Let $A \in U(A)$ be such that $\mu^i_A = 0$ for $i = 3, \ldots, d-1$.

Then $\mu^d_A$ is a Hochschild cocycle, and represents a class $[\mu^d_A] \in \text{HH}^d(A, A[2-d])$.

Proof: The $(d+1)$th Aco-equation involves $\mu^i_A$, $i = 3, 2, \ldots, d$. Since all but $\mu^2_A, \mu^d_A$ vanish, the equation reduces to the condition that $\mu^d_A$ is closed.

Define: In the situation above, we call $[\mu^d_A]$ the order $d$ obstruction class. Then we have

Prop: If $[\mu^d_A] = 0$, there is a Gauge transformation $(G^i)$ such that $G^i A = A'$ has $\mu^i_A = 0$ for $i = 3, \ldots, d$.

Consider the case where $\text{HH}^d(A, A[2-d]) = 0$ for all $d \geq 3$. Then $U(A)/Z(A)$ reduces to a point, represented by the formal Aco-structure. In this case $A$ is said to be intrinsically formal.

In the case of the quiver algebra $Q$ above, we shall find that

$$\text{HH}^d(Q, Q[2-d]) = \begin{cases} 0 & d = 3 \\ \text{Sym}^4(V) & d = 4 \\ 0 & d \geq 5 \end{cases}$$

So Aco-structures are classified by an element of $\text{Sym}^4(V)$, a quartic polynomial in 2 variables.