Lecture 29  A quiver algebra from the two-torus.

Let \( R \) be our ground field.

Consider the two-torus \( T = \mathbb{R}^2 / \mathbb{Z}^2 \) \( x = dp \text{ d}x, \quad y = dz = dp + idy \)

as before. We investigate the total endomorphism algebra of a particularly chosen pair of objects.

\[
\begin{align*}
L_1 &= \xi \phi = 0 \\
L_2 &= \xi \phi = -2p \\
\end{align*}
\]

Same conventions for brane structure as in last lecture.

\[ w_1, -w_2 \in CF^0(L_1, L_2) \]
\[ w_4, w_3 \in CF^1(L_2, L_1) \]

We declare that \((\frac{x}{2}, 0)\) represents \(w_1 \in CF^0(L_1, L_2)\) and \(w_4 \in CF^1(L_2, L_1)\)
\((0, 0)\) represents \(-w_2 \in CF^0(L_1, L_2)\) and \(w_3 \in CF^1(L_2, L_1)\)

The self-floer cohomology of \(L_1\) is \(HF^*(L_1, L_1) \cong H^*(L_1; \mathbb{R})\).

To see this, we perturb \(L_1\) off itself.

\[
\begin{align*}
CF^*(L_1, L_1) &= CF^*(L_1, L_1') \\
\deg(x) &= 0, \quad \deg(y) = 1 \\
\partial y &= 0, \quad \partial x = (\partial u_1 + 1 \partial u_2)| y = (-1 + 1) y = 0 \\
\text{So} \quad HF^0(L_1, L_1) &= R_x \\
HF^1(L_1, L_1) &= R_y \\
\end{align*}
\]

Similarly, \(HF^*(L_2, L_2) \cong H^*(L_2; \mathbb{R})\).

Thus \( \bigoplus_{i,j=1}^{2} HF^*(L_i, L_j) \) is an 8-dimensional algebra.
To compute the product, we use $L_1'$:

\[ \text{HF}(L_1, L_2) \otimes \text{HF}'(L_2, L_1) \to \text{HF}'(L_1, L_1) \]
\[ \text{HF}(L_1, L_2) \otimes \text{HF}'(L_2, L_1) \to \text{HF}'(L_1, L_1) \]

These triangles show:

\[ [w_4] \cdot [w_1] = [y] \in H'(L_1; R) \]
\[ -[w_3] \cdot [w_2] = [y] \in H'(L_1; R) \]

Using perturbed copy of $L_2$, we also see:

\[ [w_1] \cdot [w_4] \notin H'(L_2; R) \]
\[ -[w_2] \cdot [w_3] \]

equal the generator determined by the orientation of $L_2$.

For

\[ \text{HF}(L_1, L_1) \otimes \text{HF}'(L_2, L_1) \to \text{HF}'(L_2, L_1) \]
\[ \text{HF}(L_1, L_2) \otimes \text{HF}'(L_1, L_2) \to \text{HF}'(L_1, L_2) \]
\[ [x] \in \text{HF}(L_1, L_1) \]

acts as identity.

Similarly, $\text{HF}(L_2, L_2)$ acts as identity.

Lastly, product with $\text{HF}'(L_i, L_i)$ is zero for degree reasons.

**Def.** Let $Q$ be the algebra defined by taking the path algebra of the quiver:

\[ \begin{array}{ccc}
    & w_1 & \\
    w_2 & & w_3 \\
    & w_4 & \\
\end{array} \]

modulo the relations:

\[ w_3w_2 + w_4w_1 = 0, \quad w_1w_4+w_2w_3=0, \quad w_3w_1=0, \quad w_4w_2=0. \]

**Proposition.** \( \bigotimes_{i,j=1}^{2} \text{HF}(L_i, L_j) \cong Q. \)
Why did we choose these objects?
1. We shall see that $L_1$ and $L_2$ generate the Fukaya category.
2. We can find the same algebra coming from coherent sheaves on an elliptic curve.

A slightly more abstract description of $Q$:
Let $V$ be a 2-dimensional $R$-vector space. Then define a category with two objects $1, 2$.
- $\text{Hom}^0(1, 2) = V$
- $\text{Hom}^*(1, 2) = V$
- $\text{Hom}^*(2, 2) = \Lambda^0(V) \oplus \Lambda^2(V)$
The composition is wedge product.

Then $Q = \bigoplus_{i,j=1}^2 \text{Hom}^*(i, j)$

Elliptic curves: let $V$ be a two-dimensional vector space.
$\text{P}(V) =$ space of lines in $V$.
An element $p \in \text{Sym}^4(Y^*)$ defines a quartic polynomial on $V$.
Assume $p$ vanishes at 4 distinct points of $\text{P}(V)$.

Let $Y_p$ be the double covering of $\text{P}(V)$ consisting of the square roots of $p$. In local coordinates $z$ on $\text{P}(V)$
\[ p(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 \]
\[ Y_p = \frac{1}{2} y^2 = p(z) \frac{1}{2} \] this is an affine curve.
More invariantly, $Y_p$ is a subvariety of the total space of the line bundle $\mathcal{O}_{\text{P}(V)}(2)$
\[ Y_p \subseteq \text{Tot}(\mathcal{O}_{\text{P}(V)}(2) \to \text{P}(V)) \]
It is a smooth genus 4 curve, and there is a 2:1 morphism $\pi: Y_p \to \text{P}(V)$ branched over the zeros of $p$. 
\( E_1 = \mathcal{O}_{\mathbb{P}(V)} \) and \( E_2 = \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \Lambda^2(V) \), live bundles on \( \mathbb{P}(V) \).

Thus \( \text{Hom}_{\mathbb{P}(V)}(E_1, E_2) = V^* \otimes \Lambda^2(V) \cong V \)
\[ \text{Ext}^d_{\mathbb{P}(V)}(E_1, E_2) = 0. \]

Pulling back to \( Y_p \) and using \( \pi_* \mathcal{O}_{Y_p} \cong \mathcal{O}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(-2) \)
\[ \cong \mathcal{O}_{\mathbb{P}(V)} \otimes (\mathcal{O}_{\mathbb{P}(V)} \otimes \Lambda^2(V)) \]
\[ \text{Ext}^*_p(\pi^*E_1, \pi^*E_2) \cong H^*(Y_p, \pi^*E_1 \otimes \pi^*E_2) \]
\[ \cong H^*(Y_p, \pi^*(E_1 \otimes E_2)) \]
\[ \cong H^*(\mathbb{P}(V), (E_1 \otimes E_2) \otimes \pi_* \mathcal{O}_{Y_p}) \] projection formula
\[ \cong H^*(\mathbb{P}(V), E_1 \otimes E_2) \oplus H^*(\mathbb{P}(V), E_1 \otimes E_2 \otimes \mathcal{O}_{\mathbb{P}(V)}) \otimes \Lambda^2(V) \]
\[ \cong \text{Ext}^{5-k}_{\mathbb{P}(V)}(E_1, E_2) \oplus \text{Ext}^{1-k}_{\mathbb{P}(V)}(E_2, E_1)^* \otimes \Lambda^2(V). \]
\[ \cong V \text{ in degree 0}. \]

Swapping roles of \( E_1 \) and \( E_2 \), \( \text{Ext}^d_{\mathbb{P}(V)}(\pi^*E_2, \pi^*E_1) = V^* \otimes \Lambda^2(V) \cong V \)

And \( \text{Ext}^*_p(\pi^*E_1, \pi^*E_1) = \Lambda^0(Y) \oplus \Lambda^2(V) \)
\[ \text{deg} 0 \quad \text{deg} 1 \]

Proposition
\[ \text{Ext}^*_p(\pi^*E_1 \oplus \pi^*E_2, \pi^*E_1 \oplus \pi^*E_2) \cong \mathbb{Q}. \]

So the same algebra appears here!

Now there is actually a non-trivial A_\infty-structure in both cases (\( T \) and \( Y_p \)). In the case of \( Y_p \), the equivalence class of the A_\infty-structure depends on \( p \). The next task is to understand this dependence.