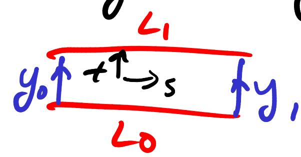
Lecture 27: Fukaya categories away from characteristic 2

Let $\mathbb{K}$ be a field and let V be a 1-dimensional real vector space. We define the $\mathbb{K}$-normalization of V to be the $\mathbb{K}$-vector space $|V|_{\mathbb{K}}$ generated by the two orientations of V, modulo the relation that their sum is zero. If $c: V_1 \to V_2$ is an isomorphism, it induces an iso $|c|_{\mathbb{K}} : |V_1|_{\mathbb{K}} \to |V_2|_{\mathbb{K}}$.

Let $L_i^{\#} = (L_i, \alpha_i^{\#}, P_i^{\#})$ $i=0,1$ be two Lagrangian branes in (M, ω, η^2). Pick a regular Floer datum (H,J). Any $y \in \mathcal{C}(L_0, L_1)$ then has an associated index $i(y)$ (from gradings) and orientation line $o(y)$ (from brane structure).

We define
$$CF^k(L_0^{\#}, L_1^{\#}) = \bigoplus_{i(y)=k} |o(y)|_{\mathbb{K}}$$
to be the degree k Floer cochains over $\mathbb{K}$.

We define $\partial : CF^k(L_0^{\#}, L_1^{\#}) \to CF^{k+1}(L_0^{\#}, L_1^{\#})$ by counting strips modulo translation: $\mathcal{M}_z(y_0, y_1) =$ strips



$$\mathcal{M}_z^*(y_0, y_1) = \mathcal{M}_z(y_0, y_1)/\mathbb{R}$$

There is an exact sequence $0 \to \mathbb{R} \to T\mathcal{M}_z(y_0,y_1)_u \to T\mathcal{M}_z^*(y_0,y_1) \to 0$

Convention: the generator of $\mathbb{R}$ corresponds to translation in positive s-direction

thus we get an isomorphism $\Lambda^{top} T\mathcal{M}_z(y_0,y_1)_u \cong \Lambda^{top} \mathcal{M}_z^*(y_0,y_1)$

We also have a canonical iso $\Lambda^{top} T\mathcal{M}_z(y_0,y_1) \cong o(y_0) \otimes o(y_1)^{\vee}$ and at an isolated solution, canonical iso $\Lambda^{top} \mathcal{M}_z^*(y_0,y_1) \cong \mathbb{R}$.

Putting it together, we get associated to $u \in \mathcal{M}_Z^*(y_0, y_1)$ an isomorphism $C_u : o(y_1) \to o(y_0)$

Let $|C_u|_{\mathbb{K}} : |o(y_1)|_{\mathbb{K}} \to |o(y_0)|_{\mathbb{K}}$ be its $\mathbb{K}$-normalization

Then we define: $$\partial^{y_0, y_1} = \sum_{u \in \mathcal{M}_Z^*(y_0, y_1)} |C_u|_{\mathbb{K}} : |o(y_1)|_{\mathbb{K}} \to |o(y_0)|_{\mathbb{K}}$$

and this operator is the y_1-to-y_0 component of ∂.

Now, for $\mu^d : CF(L_{d-1}^\#, L_d^\#) \otimes \cdots \otimes CF(L_0^\#, L_1^\#) \to CF(L_0^\#, L_d^\#)[2-d]$ we count $\mathcal{M}_S(y_0, y_1, \ldots, y_d)$ where $S \to \mathcal{R}^{d+1}$ is the family of all disks.

We have canonical isomorphisms
$$\Lambda^{top} T\mathcal{M}_S(y_0, y_1, \ldots, y_d)_{(r,u)} \cong (\Lambda^{top} T\mathcal{R}^{d+1})_r \otimes o(y_0) \otimes o(y_1)^\vee \otimes \cdots \otimes o(y_d)^\vee$$

and so at an isolated point, an isomorphism

$$o(y_d) \otimes \cdots o(y_1) \to (\Lambda^{top} T\mathcal{R}^{d+1})_r \otimes o(y_0)$$

Now we choose "by hand" an orientation of $\mathcal{R}^{d+1}$
$$\mathcal{R}^{d+1} = \mathrm{Conf}_{d+1}(\partial D)/\mathrm{Aut}(D)$$

There is an embedding $\mathcal{R}^{d+1} \hookrightarrow (\partial D)^{d-2}$ by fixing z_0, z_1, z_2 on ∂D and mapping to $(z_3, \ldots, z_d)$.
Orient ∂D as the boundary of D, and pull back the product orientation on $(\partial D)^{d-2}$.

With this choice, we have, for each isolated $u \in \mathcal{M}_{\mathcal{S}}(y_0, y_1, \dots, y_d)$, an isomorphism

$$c_u : o(y_d) \otimes \cdots \otimes o(y_1) \longrightarrow o(y_0)$$

Then define $(y_d, \dots, y_1)$-to-y_0 component of μ^d to be

$$\sum_{u \in \mathcal{M}_{\mathcal{S}}(y_0, y_1, \dots, y_d)} |c_u|_{\mathbb{K}} \; : \; |o(y_d)|_{\mathbb{K}} \otimes \cdots \otimes |o(y_1)|_{\mathbb{K}} \longrightarrow |o(y_0)|_{\mathbb{K}}$$

<u>Theorem</u>: this forms an A_∞-structure, i.e.

$$\sum_{m,n} (-1)^* \mu^{d-m+1}(a_d, \dots, a_{n+m+1}, \mu^m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1) = 0$$

where $* = \sum_{i=1}^{n} (\deg(a_i) - 1)$.

Over what we have previously said, one needs to check how the orientation conventions above behave under gluing.