lecture 24  Indices of graded Lagrangian intersection

Last couple of lectures: Lagrangian Grassmannian, Maslov classes, graded Lagrangians.

We now seek to define a grading on $CF((L_0, x_0^*), (L_1, x_1^*))$, where $(L_i, x_i^*)$ are graded Lagrangians in $(M, \omega, \gamma^2)$.

Need a bit more linear algebra of Lagrangian Grassmannian $Gr(V)$ where $(V, \omega_V)$ is a symplectic vector space.

**Crossing form of a pair of paths** Let $(\lambda_0, \lambda_1)$ be two paths $[0,1] \to Gr(V)$

For generic $s \in [0,1]$ and generic paths, $\lambda_0(s) \cap \lambda_1(s) = \emptyset$.

But for some values of $s$, $\lambda_0(s)$ and $\lambda_1(s)$ fail to be transverse.

At such a point, we define a quadratic form on $\lambda_0(s) \cap \lambda_1(s)$ as follows:

Choose continuous families of linear maps $\phi_{k, r, s} : \lambda_k(s) \to \lambda_k(r)$ for $k = 0, 1$ and $|r - s|$ small such that $\phi_{k, s, s} = Id$.

Then define $q_{\lambda_0, \lambda_1}(s)(V) = -\left(\frac{d}{dt}\right)_{t = s} \omega_V(\phi_{0, r, s}(V), \phi_{1, r, s}(V))$ for $V \in \lambda_0(s) \cap \lambda_1(s)$.

It is independent of $\phi_{k, r, s}$.

For generic paths, $\lambda_0(s) \cap \lambda_1(s)$ is at most one dimensional, and $q_{\lambda_0, \lambda_1}(s)$ is non-zero when the dimension is one; it can be either positive or negative in this case.
Now let $\mathcal{P}^- \text{Gr}(V)$ be the space of paths $\lambda : [0, 1] \rightarrow \text{Gr}(V)$ such that:

* $\lambda(0) \neq \lambda(1)$

* the pair $(\lambda, \lambda(1))$, where the second component denotes the constant path at $\lambda(1)$, has negative definite crossing form at $s = 1$.

We associate an index $I(\lambda)$ to $\lambda \in \mathcal{P}^- \text{Gr}(V)$ as

$$I(\lambda) = \sum_{0 < s < 1} \text{sign}(\nu_{\lambda, \lambda(s)}(s))$$

where $\nu_{\lambda, \lambda(s)}(s)$ is the crossing form at $\lambda(s) \cap \lambda(s)$.

(we can assume $\lambda$ is generic in $\mathcal{P}^- \text{Gr}(V)$ so $\text{sign}(\nu_{\lambda, \lambda(s)}(s)) = \pm 1$ at each crossing.)

**Graded Lagrangian Grassmannian:** We have $\text{Gr}(V) \cong U(n)/O(n) \xrightarrow{\text{det}^2} S^1$

Let $\text{Gr}^\#(V)$ be the covering space defined via pull-back of $\mathbb{R} \rightarrow S^1$

$$\text{Gr}^\#(V) \rightarrow \mathbb{R}$$

$$\pi \downarrow \downarrow$$

$$\text{Gr}(V) \rightarrow S^1$$

Call this the **Graded Lagrangian Grassmannian**.

Let $\Lambda_0^\#, \Lambda_1^\# \in \text{Gr}^\#(V)$ be a pair of graded Lagrangian subspaces.

We associate an absolute index $i(\Lambda_0^\#, \Lambda_1^\#)$ to this pair:

$$i(\Lambda_0^\#, \Lambda_1^\#) = I(\pi \circ \lambda^\#),$$

where $\lambda^\#: [0, 1] \rightarrow \text{Gr}^\#(V)$ is a path such that $\lambda^\#(0) = \Lambda_0^\#$, $\lambda^\#(1) = \Lambda_1^\#$, and $\pi \circ \lambda^\#: [0, 1] \rightarrow \text{Gr}(V)$ lies in $\mathcal{P}^- \text{Gr}(V)$. 


Now suppose that $(M,\omega)$ is a symplectic manifold and \(\eta^2 : (\Lambda^0 TM)^{\otimes 2} \to \mathbb{C}\) is a trivialization.

Let \(L_0, L_1\) be two transversely intersecting Lagrangian submanifolds. We get maps

\[
\begin{array}{ccc}
L_k & \rightarrow & \text{Gr}(TM) \\
\alpha_{L_k} & & \downarrow \\
& & S^1
\end{array}
\]

and we suppose \(\mu_{L_k} = [\alpha_{L_k}] = 0\) in \(H^1(L_k; \mathbb{Z})\).

We pick lifts \(\alpha_k^\# : L_k \rightarrow S^1\), so \((L_k, \alpha_k^\#)\) is a graded Lagrangian submanifold.

The choice of trivialization \(\eta^2\) allows us to construct a covering space

\[
\begin{array}{ccc}
\text{Gr}^\#(TM) & \rightarrow & \mathbb{R} \\
\downarrow & & \downarrow \\
\text{Gr}(TM) & \rightarrow & S^1
\end{array}
\]

which is a fiberwise \(\mathbb{Z}\)-fold cover of \(\text{Gr}(TM)\).

Then the choice of grading \(\alpha_k^\#\) on \(L_k\) amounts to a lift

\[
\begin{array}{ccc}
L_k & \rightarrow & \text{Gr}^\#(TM) \\
\alpha_{L_k} & & \downarrow \\
& & \text{Gr}(TM)
\end{array}
\]

At a point \(p \in L_0 \cap L_1\), we get a pair of graded Lagrangian subspaces \(j_{0\#}(p), j_{1\#}(p) \in \text{Gr}^\#(T_pM)\)

and we define the index of \(p\) as \(i(p) = i(j_{0\#}(p), j_{1\#}(p))\).
Examples in dimension 2:

\[ L_1, \alpha_1^\# \in (0,1) \]

\[ L_0, \alpha_0^\# = 0 \]

The short clockwise rotation from \( L_0 \) to \( L_1 \) lies in \( \mathbb{P} \text{-Gr}(T_pM) \)

But it doesn't lift to \( \text{Gr}^*(T_pM) \) properly.

The path that rotates \( L_0 \) counter-clockwise past \( L_1 \) and then back lies in \( \mathbb{P} \text{-Gr}(T_pM) \) and does lift properly. It has one positive crossing, so \( i(p) = 1 \)

If we were to take \( \alpha_1^\# \in (-1,0) \), we would have computed \( i(p) = 0 \) instead.

On \( T^*S^1 \)

Gray lines = line field

Can take \( \alpha_0^\# = 0 \), \( \alpha_1^\# \in (-1,1) \)

then

\[ i(x) = 0 \]
\[ i(y) = 1 \]

Next: 0 has degree 1: