Lecture 19  Examples of Floer cohomology.

(Chw(p) = 2)

A some of computable examples comes from target surfaces, i.e. dim M = 2, a symplectic form is just an area form, and any 1-dim submanifold L \subset M is Lagrangian.

\[ M = T^*S^1 = \mathbb{R} \times S^1 \]  

We can take \[ L = \mathbb{R} \times S^1 \]

To compute \( HF(L,L) \), we need to choose a perturbation datum. In particular, we need to choose a function \( H : M \rightarrow \mathbb{R} \) such that \( \varphi_H(L) \cap L \).

Let \( (r, \theta) \) be coord. ds on \( T^*S^1 = \mathbb{R} \times S^1 \) such that \( w = d\theta dr \) is the symplectic form.

Let \( H(r, \theta) = h(\theta) \), where \( h(\theta) : S^1 \rightarrow \mathbb{R} \) is a function with a unique nondegenerate \( (h' \neq 0) \) maximum and a unique nondegenerate minimum. Then \( dH = h'(\theta) d\theta \) and

\[ w(-, X_H) = dH \]

means

\[ d\theta \cdot dr(X_H) - dr d\theta(X_H) = h'(\theta) d\theta \]

so \( d\theta(X_H) = 0 \) and \( -dr(X_H) = h'(\theta) \)

so \( X_H = \frac{h'(\theta)}{h'} \frac{\partial}{\partial r} \)

and \( \varphi'_H(r, \theta) = (r + h'(\theta), \theta) \)

so \( \varphi'_H \) shifts \( r \)-coordinate by an amount that depends on \( \theta \).
To make it a little more clear, we can unroll the cylinder

The generators of $\text{CF}(L,L)$ (for this perturbation) are the two intersection points of $\varphi_h'(L)$ and $L$.

To find $\partial$, we must count strips joining these two points; these strips are visible in the perturbed picture

Since the bottom boundary corresponds to $\varphi_h'(L)$, and the map is read right to left, there are two strips connecting $x \to y$, so

$\partial(x) = 2y = 0 \pmod{2}$

and $\partial(y) = 0 \pmod{y}$

So $\partial = 0$ (and $\partial^2 = 0$)

The Floer cohomology is $\text{HF}(L,L) = \text{CF}(L,L) = \{x,y\}$

is a rank 2 vector space over $\mathbb{K}$.

One could use different perturbations: e.g.

Then $\partial(x_1) = y_1 + y_2$

$\partial(x_2) = y_1 + y_2$

$\partial(y_1) = 0 = \partial(y_2)$

$\ker(\partial) = \langle y_1, y_2, x_1 - x_2 \rangle$ \hspace{1cm} $\text{rk} = 3$

$\text{Im}(\partial) = \langle y_1 + y_2 \rangle$ \hspace{1cm} $\text{rk} = 1$

$\text{HF}(L,L) \hspace{1cm} \text{rk} = 2$
An example where $\exists^2 \neq 0$: $M = \mathbb{R}^2 \setminus \{0\}$

$\exists(x) = y$
$\exists(y) = x$
$\exists^2 = \text{Id}$

Issue: $L_1$ bounds a disk