We have so far discussed how one may show $\partial \circ \partial = 0$ and $\partial \circ \overline{\partial} = \overline{\partial} \circ \partial$.

Next we claim that

(a) $\overline{\partial}$ is independent of the choice of complex structure and perturbation data (relative Fredholm data on striplike ends.)

(b) The gluing axiom holds

(c) $HF^0(L_0, L_1)$ is independent of choice of Floer data up to isomorphism.

For all of these we must use parametrized moduli spaces

(A key ingredient of chain level structures such as the Fukaya category)

Let $R$ be a manifold, possibly with boundary and corners. Let $S = \{S_t\}$ be a family of pointed boundary Riemann surfaces parametrized by $t \in R$ (complex structure varies.) Pick a set of Lagrangian labels, and let $(K_r, J_r)$ be a family of perturbation data. Assume Floer data fixed on the ends. Let

$$M_{S_r}(\delta y_s^3) = \{ \text{ solutions } u : S_r \to M \text{ for } (K_r, J_r) \}$$

and

$$M_S(\delta y_s^3) = \bigcup_{r \in R} M_{S_r}(\delta y_s^3)$$

This is the parametrized moduli space for the family $(S_r, K_r, J_r)$
Under appropriate hypotheses, $M_g(\mathfrak{P}yS^3)$ can be given the structure of a manifold, and it admits a Grothendieck compactification $\overline{M}_g(\mathfrak{P}yS^3)$.

(a) Choose a family $(S_r, K_r, J_r) \in [0,1]$ that interpolates between the two choices $(S_0, K_0, J_0)$, $(S_1, K_1, J_1)$.

This is possible because the space of choices is contractible.

Then $M_g(\mathfrak{P}yS^3)$ has boundary points of 3 types:

1. $S_0 \rightarrow C\Phi_{S_0}$

   Use $M_g(\mathfrak{P}yS^3)$ to define an operator $P$, and then we have

   \[ C\Phi_{S_0} - C\Phi_{S_0} = \Delta P + P \Delta \]

   $C\Phi_{S_0}$ and $C\Phi_{S_0}$ induce same map on cohomology.

2. $S_r \in M_g(\mathfrak{P}yS^3) \ni \exists \epsilon$

   \[ \begin{cases} \text{contributes to } P\Delta \text{ or } \Delta P \text{ depending on whether strip is at incoming/outgoing puncture.} 
\end{cases} \]
(b) Take $S_0$ and $S_1$, and glue a strip like end of $S_0$ to one of $S_1$.

If we choose perturbation data very carefully, we can prove gluing at chain level, but only for this particular choice. By (a), the result holds in homology for any choice.

(c) This is now a formal consequence. Take $(L_0, L_1)$ and $(H_0, J_0)$ and $(H_1, J_1)$ two choices of Floer data.

Pick a perturbing datum $(K_{01}, J_{01})$ on $\mathbb{R} \times [0, 1]$ interpolating between them, and choose $(K_{10}, J_{10})$ interpolating in the other direction.

Then $\text{C}_\Phi(z, K_{01}, J_{01}) : \text{CF}^\text{fr}(L_0, L_1, H_0, J_0) \rightarrow \text{CF}^\text{fr}(L_0, L_1, H_1, J_1)$

and similarly for $(z, K_{10}, J_{10})$

By gluing $(\Phi z, K_{01}, J_{01}) \circ (z, K_{01}, J_{01}) = \text{Id} : \text{HF}^\text{fr}(L_0, L_1, H_0, J_0)$

and similarly

$(\Phi z, K_{01}, J_{01}) \circ (z, K_{01}, J_{01}) = \text{Id} : \text{HF}^\text{fr}(L_0, L_1, H_1, J_1)$

so $\text{HF}^\text{fr}(L_0, L_1, H_0, J_0) \xrightarrow{\Phi (z, K_{01}, J_{01})} \text{HF}^\text{fr}(L_0, L_1, H_1, J_1)$

are isomorphisms.